Local approximations and intrinsic characterizations of spaces of smooth functions on regular subsets of $\mathbf{R}^n$

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Abstract

We give an intrinsic characterization of the restrictions of Sobolev $W_p^k(\mathbf{R}^n)$, Triebel-Lizorkin $F_{pq}^s(\mathbf{R}^n)$ and Besov $B_{pq}^s(\mathbf{R}^n)$ spaces to regular subsets of $\mathbf{R}^n$ via sharp maximal functions and local approximations.

1. Main definitions and results.

The purpose of this paper is to study the problem of extendability of functions defined on measurable subsets of $\mathbf{R}^n$ to functions defined on the whole space and satisfying certain smoothness conditions.

We will consider three kinds of spaces of smooth functions defined on $\mathbf{R}^n$. First we deal with classical Sobolev spaces, see e.g. Maz'ja [27]. We recall that, given an open set $\Omega \subset \mathbf{R}^n$, $k \in \mathbf{N}$ and $p \in [1, \infty]$, the Sobolev space $W_p^k(\Omega)$ consists of all functions $f \in L_{1,loc}(\Omega)$ whose distributional partial derivatives on Ω of all orders up to k belong to $L_p(\Omega)$. $W_p^k(\Omega)$ is normed by $\|f\|_{W_p^k(\Omega)} := \sum\{\|D^\alpha f\|_{L_p(\Omega)} : |\alpha| \le k\}$.

There is an extensive literature devoted to describing the restrictions of Sobolev functions to different classes of subsets of $\mathbf{R}^n$. (We refer the reader to [27, 28, 3, 4, 12, 22, 16, 31] and references therein for numerous results and techniques in this direction.) Let us recall some of these results. Calderón [14] showed that, if Ω is a Lipschitz domain in $\mathbf{R}^n$ and $1 < p < \infty$, then $W_p^k(\mathbf{R}^n)|_\Omega = W_p^k(\Omega)$. Stein [38] extended this result for $p = 1, \infty$ and Jones [24] showed that the same isomorphism also holds for every $(\varepsilon, \delta)-$domain and $1 \le p \le \infty$.

Here as usual, for any Banach space $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ of measurable functions defined on $\mathbf{R}^n$ and a measurable set $S \subset \mathbf{R}^n$ of positive Lebesgue measure, we let $\mathcal{A}|_S$ denote the

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restriction of $\mathcal{A}$ to S, i.e., the Banach space

$$\mathcal{A}|_S := \{f : S \to \mathbf{R} \ : \text{there is} \ \ F \in \mathcal{A} \ \text{ such that } \ F|_S = f \quad \text{a. e.}\}.$$

We call $\mathcal{A}|_S$ the *restriction space* or the *trace space* of $\mathcal{A}$ to S. $\mathcal{A}|_S$ is equipped with the standard quotient space norm

$$\|f\|_{\mathcal{A}|_S} := \inf\{\|F\|_{\mathcal{A}} : F \in \mathcal{A}, F|_S = f \ \text{ a. e.}\}.$$

Our aim is to extend these results to the case of so-called *regular subsets of* $\mathbf{R}^n$. We define these sets as follows.

Definition 1.1 *A measurable set $S \subset \mathbf{R}^n$ is said to be regular if there are constants $\theta_S \geq 1$ and $\delta_S > 0$ such that, for every cube Q with center in S and with diameter* $\operatorname{diam} Q \leq \delta_S$*,*

$$|Q| \leq \theta_S |Q \cap S|.$$

Here $|A|$ stands for the Lebesgue measure of a measurable set $A \subset \mathbf{R}^n$. We will also assume that all cubes in this paper are closed and have sides which are parallel to the coordinate axes. We let $Q(x, r)$ denote the cube in $\mathbf{R}^n$ centered at x with side length $2r$.

Regular subsets of $\mathbf{R}^n$ are often called Ahlfors n-regular or n-sets [25]. Cantor-like sets and Sierpiński type gaskets (or carpets) of positive Lebesgue measure are examples of non-trivial regular subsets of $\mathbf{R}^n$.

We observe that the interior of a regular set can be empty (as, for instance, for a Cantor-like set or for a Sierpiński type gasket). Thus, to give a constructive characterization of the restrictions of Sobolev functions to regular sets, we need an equivalent definition of the Sobolev spaces which does not use the notion of derivatives.

There are several known ways of defining Sobolev spaces which do not use derivatives. In this paper our point of departure will be a characterization of Sobolev spaces due to Calderón. In [13] (see also [15]) Calderón characterizes the Sobolev spaces $W_p^k(\mathbf{R}^n)$ in terms of L_p-properties of sharp maximal functions.

Before we recall Calderón's result we need to fix some notation. Let $\mathcal{P}_k = \mathcal{P}_k(\mathbf{R}^n)$, $k \geq 0$, denote the family of all polynomials on $\mathbf{R}^n$ of degree less than or equal to k. We also put $\mathcal{P}_{-1} := \{0\}$. Given $f \in L_{u,loc}(\mathbf{R}^n)$, $0 < u \leq \infty$, and a cube Q, we let $\mathcal{E}_k(f; Q)_{L_u}$ denote the *normalized local best approximation* of f on Q in L_u-norm by polynomials of degree at most $k - 1$, see Brudnyi [7]. More explicitly, we define

$$\mathcal{E}_k(f; Q)_{L_u} := |Q|^{-\frac{1}{u}} \inf_{P \in \mathcal{P}_{k-1}} \|f - P\|_{L_u(Q)} = \inf_{P \in \mathcal{P}_{k-1}} \left(\frac{1}{|Q|} \int_Q |f - P|^u \, dx \right)^{\frac{1}{u}}. \tag{1.1}$$

In the literature $\mathcal{E}_k(f; Q)_{L_u}$ is also sometimes called the *local oscillation* of f, see e.g. Triebel [40]. This quantity is the main object of the theory of *local polynomial approximations* which provides a unified framework for the description of a large family of spaces of smooth functions. We refer the reader to Brudnyi [5]-[10] for the main ideas and results in local approximation theory; see also Section 5 for additional information and remarks related to this theory.

Given $\alpha > 0$ and a locally integrable function f on $\mathbf{R}^n$, we define its *fractional sharp maximal function* $f^\sharp_\alpha$ by letting

$$f^\sharp_\alpha(x) := \sup_{r>0} r^{-\alpha}\, \mathcal{E}_k(f; Q(x,r))_{L_1}. \tag{1.2}$$

Here $k := -[-\alpha]$ is the greatest integer strictly less than $\alpha + 1$.

In [13] Calderón proved that, for $1 < p \le \infty$, a function f is in $W^k_p(\mathbf{R}^n)$, if and only if f and $f^\sharp_k$ are both in $L_p(\mathbf{R}^n)$. Moreover, up to constants depending only on n, k and p the following equivalence,

$$\|f\|_{W^k_p(\mathbf{R}^n)} \approx \|f\|_{L_p(\mathbf{R}^n)} + \|f^\sharp_k\|_{L_p(\mathbf{R}^n)}, \tag{1.3}$$

holds.

This characterization motivates the following definition. Given $u > 0$, a function $f \in L_{u,loc}(S)$, and a cube Q whose center is in S, we let $\mathcal{E}_k(f;Q)_{L_u(S)}$ denote the normalized best approximation of f on Q in $L_u(S)$-norm:

$$\mathcal{E}_k(f;Q)_{L_u(S)} := |Q|^{-\frac{1}{u}} \inf_{P\in\mathcal{P}_k} \|f-P\|_{L_u(Q\cap S)} = \inf_{P\in\mathcal{P}_{k-1}} \left(\frac{1}{|Q|}\int_{Q\cap S} |f-P|^u\, dx\right)^{\frac{1}{u}}. \tag{1.4}$$

By $f^\sharp_{\alpha,S}$, we denote the fractional sharp maximal function of f on S,

$$f^\sharp_{\alpha,S}(x) := \sup_{r>0} r^{-\alpha}\, \mathcal{E}_k(f; Q(x,r))_{L_1(S)}, \qquad x \in S.$$

Here $k (= -[-\alpha])$ is the same as in (1.2). Thus, $f^\sharp_\alpha = f^\sharp_{\alpha,\mathbf{R}^n}$.

The first main result of the paper is the following

Theorem 1.2 *Let S be a regular subset of $\mathbf{R}^n$. Then a function $f \in L_p(S)$, $1 < p \le \infty$, can be extended to a function $F \in W^k_p(\mathbf{R}^n)$ if and only if*

$$f^\sharp_{k,S} := \sup_{r>0} r^{-k}\, \mathcal{E}_k(f; Q(\cdot,r))_{L_1(S)} \in L_p(S).$$

In addition,

$$\|f\|_{W^k_p(\mathbf{R}^n)|_S} \approx \|f\|_{L_p(S)} + \|f^\sharp_{k,S}\|_{L_p(S)} \tag{1.5}$$

with constants of equivalence depending only on n, k, p, θ_S and δ_S.

For $k = 1$ this theorem follows from a more general result proved in [37] for the case of a metric space equipped with a doubling measure.

We now turn to the second kind of spaces of smooth functions to be considered in this paper, namely the Triebel-Lizorkin spaces $F^s_{pq}(\mathbf{R}^n)$. The reader can find a detailed treatment of the theory of these spaces in the monographs [40, 41, 21, 30]. The scale $F^s_{pq}(\mathbf{R}^n)$ includes, in particular, the Bessel potential spaces $H^s_p(\mathbf{R}^n) = F^s_{p,2}, 1 < p < \infty$, ([40], p. 11). These spaces which are also referred to in the literature as

fractional Sobolev spaces are generalizations of the Sobolev spaces in the following sense: $H^k_p(\mathbf{R}^n) = W^k_p(\mathbf{R}^n)$ whenever $k \in \mathbf{N}$ and $1 < p < \infty$.

Among the various equivalent definitions of Triebel-Lizorkin spaces, the most useful one for us is expressed in terms of local polynomial approximations:

Given $0 < s < k$, $1 < p < \infty$, $1 \le q \le \infty$, a function $f \in L_{1,\,loc}(\mathbf{R}^n)$ and $x \in \mathbf{R}^n$, we put

$$g(x) := \left(\int_0^1 \left(\frac{\mathcal{E}_k(f; Q(x,t))_{L_1}}{t^s}\right)^q \frac{dt}{t}\right)^{\frac{1}{q}}$$

for $q < \infty$ and $g(x) := \sup\{t^{-s}\,\mathcal{E}_k(f; Q(x,t))_{L_1} : \ t > 0\}$ for $q = \infty$. Then $f \in F^s_{pq}(\mathbf{R}^n)$ if and only if f and g are both in $L^p(\mathbf{R}^n)$. Moreover,

$$\|f\|_{F^s_{pq}(\mathbf{R}^n)} \approx \|f\|_{L_p(\mathbf{R}^n)} + \|g\|_{L_p(\mathbf{R}^n)} \tag{1.6}$$

with constants depending only on n, s, p, q and k. This description is due to Dorronsoro [19, 20], Seeger [34] and Triebel [39]; see also [40], p. 51, and references therein for a detailed history of the problem.

The second main result of the paper, Theorem 1.3, states that for a *regular* subset $S \subset \mathbf{R}^n$, the trace space $F^s_{pq}(\mathbf{R}^n)|_S$ can be characterized in a way which is analogous to the preceding definition , i.e., in terms of local approximations of functions taken on the set S instead of on $\mathbf{R}^n$.

Theorem 1.3 *Let S be a regular subset of $\mathbf{R}^n$, $0 < s < k$, $1 < p < \infty$, and $1 \le q \le \infty$. Then a function $f \in L_p(S)$ can be extended to a function $F \in F^s_{pq}(\mathbf{R}^n)$ if and only if*

$$\left(\int_0^1 \left(\frac{\mathcal{E}_k(f; Q(\cdot,t))_{L_1(S)}}{t^s}\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \in L_p(S)$$

(with usual modification for $q = \infty$). In addition,

$$\|f\|_{F^s_{pq}(\mathbf{R}^n)|_S} \approx \|f\|_{L^p(S)} + \left\|\left(\int_0^1 \left(\frac{\mathcal{E}_k(f; Q(\cdot,t))_{L_1(S)}}{t^s}\right)^q \frac{dt}{t}\right)^{\frac{1}{q}}\right\|_{L_p(S)} \tag{1.7}$$

with constants of equivalence depending only on $n, \theta_S, \delta_S, s, p, q$ and k.

Observe that for Lipschitz domains in $\mathbf{R}^n$ an intrinsic characterization of traces of F-spaces was given by Kalyabin [26]. For (ε, δ)-domains such a characterization is due to Seeger [34]; see also Triebel [39].

Let us note two particular cases of Theorem 1.3.

Remark 1.4 *Recall that $W^k_p(\mathbf{R}^n) = F^k_{p,2}$, $1 < p < \infty$. This and Theorem 1.3 imply one more intrinsic characterization of the restrictions of Sobolev functions to regular subsets, cf. (1.5): for every $1 < p < \infty$*

$$W^k_p(\mathbf{R}^n)|_S = \{f \in L_p(S) : \ \left(\int_0^1 \left(\frac{\mathcal{E}_k(f; Q(\cdot,t))_{L_1(S)}}{t^s}\right)^2 \frac{dt}{t}\right)^{\frac{1}{2}} \in L_p(S)\}.$$

Remark 1.5 *The space $F^s_{p,\infty}(\mathbf{R}^n)$, $s>0, 1\le p\le\infty$, coincides with the space $C^s_p(\mathbf{R}^n)$ introduced by DeVore and Sharpley [17]; for non-integer s this space was independently considered by Christ [16]. (See also Triebel [40], p. 48–50 for additional remarks and comments.) We recall that $C^s_p(\mathbf{R}^n)$ consists of all functions f defined on $\mathbf{R}^n$ such that $f, f^\flat_s \in L^p(\mathbf{R}^n)$. Here $f^\flat_s$ is defined by formula (1.2) with $k=[s]+1$.*

Thus by Theorem 1.3 for every regular set S, every $0<s<k$ and $1<p<\infty$,

$$C^s_p(\mathbf{R}^n)|_S = \{f\in L^p(S):\ \sup_{0<t\le 1} t^{-s}\mathcal{E}_k(f;Q(\cdot,t))_{L_1(S)}\in L_p(S)\}.$$

Observe that Devore and Sharpley [17] obtained an intrinsic characterization of the trace space $C^s_p(\mathbf{R}^n)|_\Omega$ where Ω is a Lipschitz domain in $\mathbf{R}^n$; for (ε,δ)-domain it was done by Christ [16].

Having dealt with Sobolev and Triebel-Lizorkin spaces we now turn finally to consider Besov spaces $B^s_{pq}(\mathbf{R}^n)$. For a general theory of these spaces we refer the reader to the monographs [3, 40, 30] and references therein. See also Section 5 for definitions and a description of the Besov spaces via local approximations. This description provides the following equivalent norm on the Besov spaces: for all $1\le u\le p\le\infty$, $0<q\le\infty$, and $0<s<k$

$$\|f\|_{B^s_{pq}(\mathbf{R}^n)}\approx \|f\|_{L_p(\mathbf{R}^n)}+\left(\int_0^1\left(\frac{\|\mathcal{E}_k(f;Q(\cdot,t))_{L_u}\|_{L_p(\mathbf{R}^n)}}{t^s}\right)^q\frac{dt}{t}\right)^{\frac{1}{q}} \tag{1.8}$$

(with usual modification if $q=\infty$). Here constants of the equivalence depend only on n,s,k,p and q. This characterization (in an equivalent form, via so-called (k,p)-modulus of continuity in L_u , see (5.3)) was obtained by Brudnyi [7]; we also refer to Triebel [40], p. 51, and references therein.

Our next result, Theorem 1.6, states that, similar to Sobolev and F-spaces, a natural generalization of description (1.8) to regular sets provides an intrinsic characterization of the restrictions of Besov functions.

Theorem 1.6 *Let S be a regular subset of $\mathbf{R}^n$, $0<s<k$, $1\le u\le p\le\infty$ and $0<q\le\infty$. Then a function $f\in L_p(S)$ can be extended to a function $F\in B^s_{pq}(\mathbf{R}^n)$ if and only if*

$$\int_0^1\left(\frac{\|\mathcal{E}_k(f;Q(\cdot,t))_{L_u(S)}\|_{L_p(S)}}{t^s}\right)^q\frac{dt}{t}<\infty$$

($\sup\limits_{0<t\le1} t^{-s}\|\mathcal{E}_k(f;Q(\cdot,t))_{L_u(S)}\|_{L_p(S)}<\infty$ for $q=\infty$). In addition,

$$\|f\|_{B^s_{pq}(\mathbf{R}^n)|_S}\approx \|f\|_{L^p(S)}+\left(\int_0^1\left(\frac{\|\mathcal{E}_k(f;Q(\cdot,t))_{L_u(S)}\|_{L_p(S)}}{t^s}\right)^q\frac{dt}{t}\right)^{\frac{1}{q}}$$

(modification if $q=\infty$). Here constants of equivalence depend only on $n,\theta_S,\delta_S,s,p,q$ and k.

For intrinsic description of the Besov spaces on Lipschitz domains we refer the reader to Nikol'ski [29] and Besov [1, 2]; see also Rychkov [32]. The case of (ε, δ)-domains was independently treated by Shvartsman [36], Seeger [34] and Devore and Sharpley [18].

Proofs of Theorems 1.2, 1.3 and 1.6 are based on a modification of the Whitney extension method suggested in author's work [35], see also [36].

A crucial step of this approach is presented in Section 2. Without loss of generality we may assume that S is closed so that $\mathbf{R}^n \setminus S$ is open. By $W_S = \{Q_k\}$ we denote a Whitney decomposition of $\mathbf{R}^n \setminus S$, see e.g. [38].

We assign every cube $Q = Q(x_Q, r_Q) \in W_S$ a measurable subset $H_Q \subset S$ such that $H_Q \subset Q(x_Q, 10r_Q) \cap S$, $|Q| \le \gamma_1 |H_Q|$ whenever $\operatorname{diam} Q \le \delta_S$, and a family of sets $\mathcal{H}_S := \{H_Q : Q \in W_S\}$ has *a finite covering multiplicity* , i.e., every point $x \in S$ belongs to at most γ_2 sets of the family $\mathcal{H}_S$. Here γ_1, γ_2 are positive constants depending only on n and θ_S. We call every set $H_Q \in \mathcal{H}_S$ a *"reflected quasi-cube"* associated to the Whitney cube Q. The existence of the family $\mathcal{H}_S$ of reflected quasi-cubes is proved in Theorem 2.4.

The second step of the extension method is presented in Section 3. We associate every function $f \in L_{1,loc}(S)$, and every Whitney cube $Q \in W_S$ of $\operatorname{diam} Q \le \delta_S$, a *linear* mapping $P_Q : L_1(H_Q) \to \mathcal{P}_{k-1}$ which provides almost the best polynomial approximation of f on H_Q in the L_u-metric *for all* $1 \le u \le \infty$. Thus

$$\|f - P_Q f\|_{L_u(H_Q)} \approx \inf_{P \in \mathcal{P}_{k-1}} \|f - P\|_{L_u(H_Q)}$$

with constants depending only on n, k and θ_S. We put $P_Q f := 0$ if $\operatorname{diam} Q > \delta_S$. We define an extension $\tilde{f}$ by the formula

$$\tilde{f}(x) = (\operatorname{Ext}_{k,S} f)(x) := \sum_{Q \in W_S} \varphi_Q(x) P_Q f(x), \qquad x \in \mathbf{R}^n \setminus S, \tag{1.9}$$

and $\tilde{f}(x) := f(x), x \in S$. Here $\{\varphi_Q : Q \in W_S\}$ is a partition of unity subordinated to the Whitney decomposition W_S.

This extension construction was first used in [35, 36] to obtain a description of the restrictions of Besov functions to regular sets. (In Section 5 we present details of this approach and some main facts related to the local approximation theory.) I am very thankful to Yu. Brudnyi for the excellent suggestion that the same construction might also yield a characterization of the restriction of Triebel-Lizorkin functions to regular sets via local approximations.

We show that the extension operator $\tilde{f} = \operatorname{Ext}_{k,S}$ defined by (1.9) in some sense preserves local approximation properties of functions, see Theorem 3.6. For example, Theorem 3.11 states that for every cube $Q = Q(x, r)$ such that $x \in S$ and $r \le \delta_S/4$ and every $1 \le u \le \infty$

$$\mathcal{E}_k(\tilde{f}; Q(x, r))_{L_u} \le \gamma \mathcal{E}_k(f; Q(x, 25r))_{L_u(S)}$$

where γ is a positive constant depending only on n, k, θ_S and δ_S.

In Section 4 we study extension properties of certain generalized sharp maximal functions. These maximal functions determine both the norm in the Sobolev space and

the norm in the Triebel-Lizorkin space. Given a vector of parameters $\vec{v} := (s, k, q, u)$ where $0 \le s \le k$, $0 < q \le \infty$, $1 \le u \le \infty$ and a function $f \in L_{u,loc}(S)$, we put

$$f^{\sharp}_{\vec{v},S}(x) := \left(\int_0^\infty \left(\frac{\mathcal{E}_k(f; Q(x,t))_{L_u(S)}}{t^s} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}, \qquad x \in S,$$

(with the corresponding modification if $q = \infty$).

Theorem 4.1 presents point-wise estimates of $(\tilde{f})^{\sharp}_{\vec{v},\mathbf{R}^n}$ via the Hardy-Littlewood maximal function of f and $f^{\sharp}_{\vec{v},S}$. To formulate the result given a function h defined on S, we let $h^{\curlywedge}$ denote its extension on all of $\mathbf{R}^n$ by zero. We prove that for every $1 \le u \le \infty$, $0 \le s < k$ (or $0 \le s \le k$ if $q = \infty$) and every $x \in \mathbf{R}^n$

$$(\tilde{f})^{\sharp}_{\vec{v},\mathbf{R}^n}(x) \le \gamma\{M((f^{\sharp}_{\vec{v},S})^{\curlywedge})(x) + M_u(f^{\curlywedge})(x)\} \tag{1.10}$$

where $M_u f := (M(|f|^u))^{\frac{1}{u}}$ and $\gamma = \gamma(n, k, s, \theta_S, \delta_S)$. For instance, $f^{\sharp}_{\alpha} = f^{\sharp}_{\vec{v},\mathbf{R}^n}$ whenever $\vec{v} := (\alpha, k, \infty, 1)$, see (1.2), so that by (1.10) for every $0 \le \alpha \le k$ we have

$$(\tilde{f})^{\sharp}_{\alpha}(x) \le \gamma(M(f^{\sharp}_{\alpha,S})^{\curlywedge}(x) + Mf^{\curlywedge}(x)), \qquad x \in \mathbf{R}^n.$$

Finally, applying the Hardy-Littlewood maximal theorem to inequality (1.10) we obtain the statements of Theorem 1.2 ($\vec{v} := (k, k, \infty, 1)$) and Theorem 1.3 ($\vec{v} := (s, k, q, 1), 0 < s < k$).

In turn the proof of Theorem 1.6 is based on estimates of the modulus of continuity of the extension $\tilde{f}$ via local approximations of f on S. In Section 5 we prove that for every regular set S and every function $f \in L_p(S)$, $1 \le p \le \infty$, the modulus of continuity of order k of $\tilde{f}$ in L_p satisfies the following inequality

$$\omega_k(\tilde{f}; t)_{L_p} \le \gamma\, t^k \left\{ \left(\int_t^1 \left(\frac{\|\mathcal{E}_k(f; Q(\cdot, \tau))_{L_p(S)}\|_{L_p(S)}}{\tau^k} \right)^p \frac{d\tau}{\tau} \right)^{\frac{1}{p}} + \|f\|_{L_p(S)} \right\}. \tag{1.11}$$

Here $0 < t \le 1$ and the constant γ depends only on k, n, p, θ_S and δ_S.

Similar estimates for the quantity $\|\mathcal{E}_k(\tilde{f}; Q(\cdot, t))_{L_u}\|_{L_p(\mathbf{R}^n)}$ are given in Theorem 5.3. Using these estimates, description (1.8) and the Hardy inequality we obtain the result of Theorem 1.6.

Remark 1.7 *Observe that the operator* $\mathrm{Ext}_{k,S}$ *defined by (1.9) provides a "universal" linear continuous extension operator from* $W^k_p(\mathbf{R}^n)|_S$, $F^s_{pq}(\mathbf{R}^n)|_S$ *and* $B^s_{pq}(\mathbf{R}^n)|_S$ *into corresponding spaces on* $\mathbf{R}^n$. *(The "universality" means that, for all sufficiently large* k, *the operator* $\mathrm{Ext}_{k,S}$ *depends only on the regular set* S *and is independent of the indices of the spaces). This allows us to complement the statements of Theorems 1.2, 1.3 and 1.6 with the following assertion:*

There exists a linear extension operator mapping functions on S *to functions on* $\mathbf{R}^n$ *which is continuous from* $\mathcal{A}|_S$ *into* $\mathcal{A}$ *whenever* $\mathcal{A}$ *is any one of the following spaces:*

(i) $W^k_p(\mathbf{R}^n)$ *for* $1 < p \le \infty$,

(ii) $F^s_{pq}(\mathbf{R}^n)$ *for* $s > 0$, $1 < p < \infty$, *and* $1 \le q \le \infty$,

(iii) $B^s_{pq}(\mathbf{R}^n)$ *for* $s > 0, 1 \le p \le \infty$, $0 < q \le \infty$.

The norm of this operator is bounded by a constant which depends only on n, θ_S, δ_S and the parameters of the space $\mathcal{A}$.

For the Triebel-Lizorkin spaces the existence of a linear continuous extension operator from $F^s_{pq}(\mathbf{R}^n)|_S$ into $F^s_{pq}(\mathbf{R}^n)$ where S is an arbitrary regular set, has been proved by Rychkov [31]. For the scale of the Besov spaces this follows from a result of Shvartsman [35], see also [36].

Remark 1.8 *As we have noted above, our goal is to give a constructive intrinsic characterization of the restrictions of smooth functions to regular sets. The descriptions of trace spaces given in Theorems 1.2, 1.3 and 1.6 are not quite constructive, because they use the notion of the best local polynomial approximation of a function on a regular set, the quantity $\mathcal{E}_k(f;Q)_{L_u(S)}$, see (1.4).*

However, following an idea of Yu. Brudnyi [9], we can readily eliminate this element of nonconstructivity. In fact, using Proposition 3.4 and the regularity of S, we immediately obtain the equivalence

$$\mathcal{E}_k(f;Q(x,r))_{L_u(S)} \approx |Q|^{-\frac{1}{u}}\|f-\mathrm{Pr}_{k,Q\cap S}(f)\|_{L_u(Q\cap S)}, \qquad x\in S, \quad r\le 1.$$

Here $1\le u\le\infty$ and $\mathrm{Pr}_{k,Q\cap S}(f)\in\mathcal{P}_{k-1}$ denotes the polynomial of the best approximation of f on $Q\cap S$ in L_2-norm. Of course there are many constructive formulas for calculation of this polynomial, see e.g. (3.4). For instance, for $k=1$, one can take $\mathrm{Pr}_{1,Q\cap S}(f)$ to be the average of f over $Q\cap S$ so that in this case

$$\mathcal{E}_1(f;Q(x,r))_{L_u(S)} \approx \left(\frac{1}{|Q|}\int_{Q\cap S}\left|f-\frac{1}{|Q\cap S|}\int_{Q\cap S} f\,dy\right|^u dx\right)^{\frac{1}{u}}, \qquad x\in S, \quad r\le 1.$$

2. The Whitney covering and a family of associated "quasi-cubes".

Our notation is fairly standard. Throughout the paper $C, C_1, C_2, ...$ or $\gamma, \gamma_1, \gamma_2, ...$ will be generic positive constants which depend only on n, θ_S, δ_S and indexes of spaces (s, p, q, k, etc.). These constants can change even in a single string of estimates. The dependence of a constant on certain parameters is expressed, for example, by the notation $\gamma = \gamma(n,k,p)$. We write $A\approx B$ if there is a constant $C\ge 1$ such that $A/C\le B\le CA$.

It will be convenient for us to measure distances in $\mathbf{R}^n$ in the uniform norm

$$\|x\|_\infty := \max\{|x_i|:\ i=1,...,n\}, \quad x=(x_1,...,x_n)\in\mathbf{R}^n.$$

Thus every cube

$$Q=Q(x,r):=\{y\in\mathbf{R}^n : \|y-x\|_\infty\le r\}$$

is a "ball" in $\|\cdot\|_\infty$-norm of "radius" r centered at x. We let $x_Q := x$ denote center of Q and $r_Q := r$ its "radius". Given a constant $\lambda>0$, we let λQ denote the cube $Q(x,\lambda r)$. By Q^* we denote the cube $Q^* := \frac{9}{8}Q$.

As usual given subsets $A, B\subset\mathbf{R}^n$, we put $\operatorname{diam} A := \sup\{\|a-a'\|_\infty :\ a,a'\in A\}$ and

$$\operatorname{dist}(A,B) := \inf\{\|a-b\|_\infty :\ a\in A, b\in B\}.$$

We also set $\operatorname{dist}(x, A) := \operatorname{dist}(\{x\}, A)$ for $x \in \mathbf{R}^n$. By $\operatorname{cl}(A)$ we denote the closure of A in $\mathbf{R}^n$. Finally, χ_A denotes the characteristic function of A; we put $\chi_A \equiv 0$ if $A = \emptyset$.

The following property of regular sets is well known (see, e.g. [37]).

Lemma 2.1 *$|\operatorname{cl}(S) \setminus S| = 0$ provided S is a regular subset of $\mathbf{R}^n$.*

In the remaining part of the paper we will assume that S is a *closed* regular subset of $\mathbf{R}^n$. Since now $\mathbf{R}^n \setminus S$ is an open set, it admits a Whitney decomposition W_S, see, e.g. Stein [38]. We recall the main properties of W_S.

Theorem 2.2 *$W_S = \{Q_k\}$ is a countable family of cubes such that*

(i). $\mathbf{R}^n \setminus S = \cup\{Q : Q \in W_S\}$;

(ii). For every cube $Q \in W_S$

$$\operatorname{diam} Q \le \operatorname{dist}(Q, S) \le 4 \operatorname{diam} Q;$$

(iii). Every point of $\mathbf{R}^n \setminus S$ is covered by at most $N = N(n)$ cubes from W_S.

We also need certain additional properties of Whitney's cubes which we present in the next lemma. These properties readily follow from (i)-(iii).

Lemma 2.3 *(1). If $Q, K \in W_S$ and $Q^* \cap K^* \neq \emptyset$, then*

$$\frac{1}{4} \operatorname{diam} Q \le \operatorname{diam} K \le 4 \operatorname{diam} Q.$$

(Recall that $Q^* := \frac{9}{8}Q$.)

(2). For every cube $K \in W_S$ there are at most $N = N(n)$ cubes from the family $W_S^* := \{Q^* : Q \in W_S\}$ which intersect K^*.

Observe that the family of cubes W_S constructed in [38] satisfies conditions of Theorems 2.2 and Lemma 2.3 with respect to the Euclidean norm rather than the uniform one. However, a simple modification of this construction provides a family of Whitney's cubes which have the same properties with respect to the uniform norm.

Let us formulate the main result of the section.

Theorem 2.4 *Let S be a regular subset of $\mathbf{R}^n$. There is a family of Borel sets $\mathcal{H}_S = \{H_Q : \ Q \in W_S\}$ such that:*

(i). $H_Q \subset (10Q) \cap S, \quad Q \in W_S$;

(ii). $|Q| \le \gamma_1 |H_Q|$ whenever $Q \in W_S$ and $\operatorname{diam} Q \le \delta_S$;

(iii). $\sum\limits_{Q \in W_S} \chi_{H_Q} \le \gamma_2$.

Here γ_1, γ_2 are positive constants depending only on n and θ_S.

Proof. Let $K = Q(x_K, r_K) \in W_S$ and let $a_K \in S$ be a point nearest to x_K on S. Then by property (ii) of Theorem 2.2

$$Q(a_K, r_K) \subset 10K. \tag{2.1}$$

Given $\varepsilon, 0 < \varepsilon \le 1,$, we denote $K_\varepsilon := Q(a_K, \varepsilon r_K)$. Let $Q = Q(x_Q, r_Q)$ be a cube from W_S with $\operatorname{diam} Q \le \delta_S$. Set

$$\mathcal{A}_Q := \{K = Q(x_K, r_K) \in W_S : \ K_\varepsilon \cap Q_\varepsilon \ne \emptyset, \ r_K \le \varepsilon r_Q\}. \tag{2.2}$$

(Recall that $Q_\varepsilon := Q(a_Q, \varepsilon r_Q)$.) We define a "quasi-cube" H_Q by letting

$$H_Q := (Q_\varepsilon \cap S) \setminus (\cup\{K_\varepsilon : \ K \in \mathcal{A}_Q\}). \tag{2.3}$$

If $\operatorname{diam} Q > \delta_S$ we put $H_Q := \emptyset$.

Prove that for some $\varepsilon := \varepsilon(n, \theta_S) \le 1$ small enough the family of subsets $\mathcal{H}_S$ satisfies conditions (i)-(iii). By (2.3) and (2.1)

$$H_Q \subset Q_\varepsilon := Q(a_Q, \varepsilon r_Q) \subset Q(a_Q, r_Q) \subset 10Q. \tag{2.4}$$

In addition, by (2.3) $H_Q \subset S$ so that $H_Q \subset (10Q) \cap S$ proving property (i).

Let us prove (ii). Suppose that $Q = Q(x_Q, r_Q) \in W_S$ and $\operatorname{diam} Q \le \delta_S$. If $K \in \mathcal{A}_Q$, then by (2.2) $K_\varepsilon \cap Q_\varepsilon \ne \emptyset$ and $r_K \le \varepsilon r_Q$. Hence

$$r_{K_\varepsilon} (= \varepsilon r_K) \le \varepsilon r_{Q_\varepsilon} (= \varepsilon^2 r_Q) \le r_{Q_\varepsilon}$$

so that $a_K \in 2Q_\varepsilon$. Since $Q(a_K, r_K) \subset 10K$, see (2.1), $K \subset Q(a_K, 10r_K)$ as well. In addition, $r_K \le \varepsilon r_Q = r_{Q_\varepsilon}$ which implies $K \subset 12Q_\varepsilon$. Thus

$$\mathcal{U}_Q := \cup\{K : \ K \in \mathcal{A}_Q\} \subset 12Q_\varepsilon. \tag{2.5}$$

By property (iii) of Theorem 2.2

$$\sum_{K \in \mathcal{A}_Q} \chi_K \le \sum_{K \in W_S} \chi_K \le N = N(n),$$

so that by (2.5)

$$\sum_{K \in \mathcal{A}_Q} |\, K| = \int\limits_{\mathcal{U}_Q} \sum_{K \in \mathcal{A}_Q} \chi_K \, dx \le \int\limits_{12Q_\varepsilon} N dx = N|\, 12Q_\varepsilon| = N12^n|\, Q_\varepsilon| = C_1|\, Q_\varepsilon|.$$

On the other hand, for every $K = Q(x_K, r_K) \in \mathcal{A}_Q$ we have

$$|\, K_\varepsilon| = |\, Q(a_K, \varepsilon r_K)| = \varepsilon^n|\, Q(a_K, r_K)| = \varepsilon^n|\, K|.$$

Hence

$$|\cup\{K_\varepsilon : \ K \in \mathcal{A}_Q\}| \le \sum_{K \in \mathcal{A}_Q} |\, K_\varepsilon| = \varepsilon^n \sum_{K \in \mathcal{A}_Q} |\, K| \le C_1 \varepsilon^n |\, Q_\varepsilon|.$$

Since S is regular and $\operatorname{diam} Q_\varepsilon = \varepsilon \operatorname{diam} Q \le \delta_S$, $|\, Q_\varepsilon \cap S| \ge \theta_S^{-1}|\, Q_\varepsilon|$ so that

$$\begin{aligned} |\, H_Q| &= |\,(Q_\varepsilon \cap S| \setminus (\cup\{K_\varepsilon : \ K \in \mathcal{A}_Q\})| \\ &\ge |\, Q_\varepsilon \cap S| - |\cup\{K_\varepsilon : \ K \in \mathcal{A}_Q\}| \ge (\theta_S^{-1} - C_1\varepsilon^n)|\, Q_\varepsilon|. \end{aligned}$$

Clearly, $|\,Q_\varepsilon| = |\,Q(a_Q, \varepsilon r_Q)| = \varepsilon^n r_Q^n = \varepsilon^n |\,Q|$ so that

$$|\,H_Q| \ge (\theta_S^{-1} - C_1\varepsilon^n)\varepsilon^n |\,Q|.$$

We define ε by setting $\varepsilon := (2C_1\theta_S)^{-\frac{1}{n}}$. Then the inequality $|\,Q| \le \gamma_1 |\,H_Q|$ holds with $\gamma_1 := 4C_1\theta_S^2$ proving property (ii) of the theorem.

Let us prove (iii). Let $Q = Q(x_Q, r_Q), Q' = Q(x_{Q'}, r_{Q'}) \in W_S$ be Whitney's cubes such that $\operatorname{diam} Q$, $\operatorname{diam} Q' \le \delta_S$ and $H_Q \cap H_{Q'} \ne \emptyset$. Since $H_Q \subset Q_\varepsilon, H_{Q'} \subset Q'_\varepsilon$, we have $Q_\varepsilon \cap Q'_\varepsilon \ne \emptyset$.

On the other hand, $Q \notin \mathcal{A}_{Q'}$ and $Q' \notin \mathcal{A}_Q$, otherwise by (2.2) and (2.3) $H_Q \cap H_{Q'} = \emptyset$. Since $Q_\varepsilon \cap Q'_\varepsilon \ne \emptyset$, by definition (2.2) $r_Q > \varepsilon r_{Q'}$ and $r_{Q'} > \varepsilon r_Q$ so that $r_Q \approx r_{Q'}$. By (2.4) $Q_\varepsilon \subset 10Q$ and similarly $Q'_\varepsilon \subset 10Q'$. But $Q_\varepsilon \cap Q'_\varepsilon \ne \emptyset$ so that $10Q \cap 10Q' \ne \emptyset$ as well. Since $r_Q \approx r_{Q'}$, this imply $Q' \subset C_2Q$ for some constant $C_2 = C_2(\varepsilon) = C_2(n, \theta_S)$. Observe also that $|\,Q| \approx |\,Q'|$.

We denote

$$\mathcal{T}_Q := \{Q' \in W_S :\ H_Q \cap H_{Q'} \ne \emptyset,\ \operatorname{diam} Q' \le \delta_S\}$$

and $\mathcal{V}_Q := \cup\{Q' :\ Q' \in \mathcal{T}_Q\}$. Thus we have proved that $\mathcal{V}_Q \subset C_2Q$ and $|\,Q'| \approx |\,Q|$ for every $Q' \in \mathcal{T}_Q$.

Let $M_Q := \operatorname{card} \mathcal{T}_Q$ be the cardinality of $\mathcal{T}_Q$. Clearly, to prove (iii) it suffices to show that $M_Q \le \gamma_2$. We have

$$M_Q |\,Q| \le C \sum_{Q' \in \mathcal{T}_Q} |\,Q'| = C \int\limits_{\mathcal{V}_Q} \sum_{Q' \in \mathcal{T}_Q} \chi_{Q'}\, dx \le C \int\limits_{C_2Q} \sum_{Q' \in \mathcal{T}_Q} \chi_{Q'}\, dx.$$

By the property (iii) of Theorem 2.2

$$\sum\{\chi_{Q'} : Q' \in \mathcal{T}_Q\} \le \sum\{\chi_{Q'} : Q' \in W_S\} \le N = N(n)$$

so that

$$M_Q |\,Q| \le C \int\limits_{C_2Q} N dx = CN |\,C_2Q| = CNC_2^n |\,Q|$$

proving the required inequality $M_Q \le \gamma_2$. $\square$

3. Local approximation properties of the extension operator.

In this section we present estimates of local polynomial approximations of the extension $\tilde{f}$, see (1.9), via corresponding local approximation of a function f defined on a regular subset $S \subset \mathbf{R}^n$. We start by presenting two lemmas about properties of polynomials on subsets of $\mathbf{R}^n$.

Proposition 3.1 *(Brudnyi and Ganzburg [11]) Let A be a measurable subset of a cube Q, $|A| > 0$, $1 \le u_1, u_2 \le \infty$ and $P \in \mathcal{P}_k$. Then*

$$|Q|^{-\frac{1}{u_1}} \|P\|_{L_{u_1}(Q)} \le \gamma |A|^{-\frac{1}{u_2}} \|P\|_{L_{u_2}(A)}$$

where γ is a positive constant depending only on n, k and the ratio $|Q|/|A|$.

The proposition implies two corollaries.

Corollary 3.2 *For every subset A of a cube Q, $|A| > 0$, every $1 \le u_1, u_2 \le \infty$ and every polynomial $P \in \mathcal{P}_k$*

$$|A|^{-\frac{1}{u_1}} \|P\|_{L_{u_1}(A)} \le \gamma |A|^{-\frac{1}{u_2}} \|P\|_{L_{u_2}(A)} \tag{3.1}$$

where γ depends only on n, k and $|Q|/|A|$.

Corollary 3.3 *Let $A_i \subset Q_i$, $|A_i| > 0$, $i = 1, 2$. Suppose that $(\lambda_1 Q_1) \cap (\lambda_1 Q_2) \ne \emptyset$ and $\lambda_2^{-1} r_{Q_1} \le r_{Q_2} \le \lambda_2 r_{Q_1}$ where λ_1, λ_2 are some positive constants. Then for every $1 \le u \le \infty$ and every polynomial $P \in \mathcal{P}_k$*

$$\|P\|_{L_u(A_1)} \le \gamma \|P\|_{L_u(A_2)}$$

where γ depends only on n, k, λ_i and $|Q_i|/|A_i|$, $i = 1, 2$.

Given a function $f \in L_{u,loc}(\mathbf{R}^n)$, $1 \le u \le \infty$, and a measurable subset $A \subset \mathbf{R}^n$, we let $E_k(f; A)_{L_u}$ denote the *local best approximation* of order k of f on A in L_u-norm, see Brudnyi [7],

$$E_k(f; A)_{L_u} := \inf_{P \in \mathcal{P}_{k-1}} \|f - P\|_{L_u(A)}. \tag{3.2}$$

Thus

$$\mathcal{E}_k(f; Q)_{L_u(S)} = |Q|^{-\frac{1}{u}} E_k(f; Q \cap S)_{L_u}$$

see (1.4). We note a simple property of $\mathcal{E}_k(f; \cdot)_{L_u(S)}$ as a cube function: for every two cubes $Q_1 \subset Q_2$

$$\mathcal{E}_k(f; Q_1)_{L_u(S)} \le \left(\frac{|Q_2|}{|Q_1|} \right)^{\frac{1}{u}} \mathcal{E}_k(f; Q_2)_{L_u(S)}. \tag{3.3}$$

Proposition 3.4 *(Brudnyi [9]) Let A be a subset of a cube Q, $|A| > 0$. Then there is a linear operator $\mathrm{Pr}_{k,A} : L_1(A) \to \mathcal{P}_{k-1}$ such that for every $1 \le u \le \infty$ and every $f \in L_u(A)$*

$$\|f - \mathrm{Pr}_{k,A}(f)\|_{L_u(A)} \le \gamma E_k(f; A)_{L_u}.$$

Here $\gamma = \gamma(n, k, \frac{|Q|}{|A|})$.

Proof. Recall the construction of $\mathrm{Pr}_{k,A}$ given in [9]. We let $\{P_\beta : |\beta| \le k - 1\}$ denote an orthonormal basis in the linear space $\mathcal{P}_{k-1}$ with respect to the inner product $\langle f, g \rangle = \int_A f(x) g(x)\, dx$. We put

$$\mathrm{Pr}_{k,A}(f) := \sum_{|\beta| \le k-1} \left(\int_A P_\beta(x) f(x)\, dx \right) P_\beta. \tag{3.4}$$

Clearly, $\mathrm{Pr}_{k,A} : L_1(A) \to \mathcal{P}_{k-1}$ is a projector (i.e., $\mathrm{Pr}_{k,A}^2 = \mathrm{Pr}_{k,A}$). Estimate its operator norm in $L_u(A)$. For every $f \in L_u(A)$ we have

$$\|\mathrm{Pr}_{k,A}(f)\|_{L_u(A)} \le \sum_{|\beta| \le k-1} \left| \int_A P_\beta(x) f(x)\, dx \right| \|P_\beta\|_{L_u(A)}$$

so that by the Hölder inequality

$$\|\operatorname{Pr}_{k,A}(f)\|_{L_u(A)} \le \left(\sum_{|\beta|\le k-1} \|P_\beta\|_{L_u(A)}\|P_\beta\|_{L_{u^*}(A)}\right)\|f\|_{L_u(A)}$$

where $1/u+1/u^*=1$. But by (3.1)

$$\|P_\beta\|_{L_u(A)}\|P_\beta\|_{L_{u^*}(A)} \le \gamma^2(|A|^{\frac{1}{u}-\frac{1}{2}}\|P_\beta\|_{L_2(A)})(|A|^{\frac{1}{u^*}-\frac{1}{2}}\|P_\beta\|_{L_2(A)}) = \gamma^2$$

proving that $\|\operatorname{Pr}_{k,A}(f)\|_{L_u(A)} \le \gamma_1\|f\|_{L_u(A)}$ with $\gamma_1 = \operatorname{card}\{\beta : |\beta| \le k-1\}\,\gamma^2$ (recall that $\|P_\beta\|_{L_2(A)} = 1$ for every β). The last inequality in the standard way implies

$$\|f - \operatorname{Pr}_{k,A}(f)\|_{L_u(A)} \le (1 + \|\operatorname{Pr}_{k,A}\|)E_k(f;A)_{L_u} \le (1+\gamma_1)E_k(f;A)_{L_u}. \qquad \square$$

Proposition 3.4 and Theorem 2.4 immediately imply the following

Corollary 3.5 *Let S be a regular set and let $Q \in W_S$ be a cube with* $\operatorname{diam} Q \le \delta_S$. *There is a linear continuous operator $P_Q : L_1(H_Q) \to \mathcal{P}_{k-1}$ such that for every function $f \in L_{u,loc}(S)$, $1 \le u \le \infty$,*

$$\|f - P_Q f\|_{L_u(H_Q)} \le \gamma E_k(f; H_Q)_{L_u}.$$

Here $\gamma = \gamma(n,k,\theta_S)$.

We put

$$P_Q f = 0, \quad \text{if} \quad \operatorname{diam} Q > \delta_S. \tag{3.5}$$

Now the map $Q \to P_Q(f)$ is defined on all of the family W_S. This map gives rise a linear extension operator defined by the formula

$$\operatorname{Ext}_{k,S} f(x) := \begin{cases} f(x), & x \in S, \\ \sum\limits_{Q\in W_S} \varphi_Q(x) P_Q f(x), & x \in \mathbf{R}^n \setminus S. \end{cases} \tag{3.6}$$

Here $\Phi_S := \{\varphi_Q : Q \in W_S\}$ is a smooth partition of unity subordinated to the Whitney decomposition W_S, see, e.g. [38]. We recall that Φ_S is a family of functions defined on $\mathbf{R}^n$ which have the following properties:

(a). $0 \le \varphi_Q \le 1$ for every $Q \in W_S$;
(b). $\operatorname{supp} \varphi_Q \subset Q^*(:= \frac{9}{8}Q)$, $Q \in W_S$;
(c). $\sum\{\varphi_Q(x) :\ Q \in W_S\} = 1$ for every $x \in \mathbf{R}^n \setminus S$;
(d). for every multiindex $\beta, |\beta| \le k$ and every cube $Q \in W_S$

$$|D^\beta \varphi_Q(x)| \le C(\operatorname{diam} Q)^{-|\beta|}, \quad x \in \mathbf{R}^n,$$

where C is a constant depending only on n and k.

We turn to estimates of local approximations of the extension operator

$$\tilde{f} := \mathrm{Ext}_{k,S}\, f.$$

To formulate the main result of the section, Theorem 3.6, given $x \in \mathbf{R}^n$ and $t > 0$ we let a_x denote a point nearest to x on S (in the uniform metric). Thus $\|x - a_x\|_\infty = \mathrm{dist}(x, S)$. We put

$$r^{(x,t)} := 50 \max(80t, \mathrm{dist}(x, S)) \tag{3.7}$$

and

$$K^{(x,t)} := Q(a_x, r^{(x,t)}). \tag{3.8}$$

Theorem 3.6 *Let S be a regular subset of $\mathbf{R}^n$ and let $f \in L_{u,loc}(S)$, $1 \le u \le \infty$. Then for every $x \in \mathbf{R}^n$ and $t > 0$*

$$\mathcal{E}_k(\tilde{f}; Q(x,t))_{L_u} \le C \frac{t^k}{t^k + \mathrm{dist}(x,S)^k} \begin{cases} \mathcal{E}_k(f; K^{(x,t)})_{L_u(S)}, & r^{(x,t)} \le \delta_S, \\ \mathcal{E}_0(f; K^{(x,t)})_{L_u(S)}, & r^{(x,t)} > \delta_S. \end{cases}$$

Here $\gamma = \gamma(n, k, \theta_S, \delta_S)$.

We recall that $\mathcal{P}_{-1} := \{0\}$ so that by definition (1.4)

$$\mathcal{E}_0(f; K^{(x,t)})_{L_u(S)} := |K^{(x,t)}|^{-\frac{1}{u}} \|f\|_{L_u(K^{(x,t)} \cap S)}. \tag{3.9}$$

We will prove the theorem for the case $1 \le u < \infty$; corresponding changes for $u = \infty$ are obvious.

The proof is based on a series of auxiliary lemmas. To formulate the first of them given a cube $K \subset \mathbf{R}^n$, we define two families of Whitney's cubes:

$$\mathcal{Q}_1(K) := \{Q \in W_S : \ Q \cap K \ne \emptyset\}$$

and

$$\mathcal{Q}_2(K) := \{Q \in W_S : \exists\, Q' \in \mathcal{Q}_1(K) \quad \text{such that} \quad Q' \cap Q^* \ne \emptyset\}. \tag{3.10}$$

Lemma 3.7 *Let K be a cube centered in S. Then for every $Q \in \mathcal{Q}_2(K)$ we have* $\mathrm{diam}\, Q \le 2\, \mathrm{diam}\, K$ *and* $\|x_K - x_Q\|_\infty \le \frac{5}{2} \mathrm{diam}\, K$.

Proof. Since $Q \in \mathcal{Q}_2(K)$, there is a cube $Q' \in \mathcal{Q}_1(K)$ such that $Q' \cap Q^* \ne \emptyset$. Let $a \in Q' \cap K$. Since $x_K \in S$, by property (ii) of Theorem 2.2

$$\mathrm{diam}\, Q' \le \mathrm{dist}(Q', S) \le \mathrm{dist}(a, S) \le \|a - x_K\|_\infty \le \frac{1}{2} \mathrm{diam}\, K$$

so that $\mathrm{diam}\, Q' \le \frac{1}{2} \mathrm{diam}\, K$. But $\mathrm{diam}\, Q \le 4\, \mathrm{diam}\, Q'$, see Lemma 2.3, (1), proving that $\mathrm{diam}\, Q \le 2\, \mathrm{diam}\, K$.

Recall that $Q' \cap K \ne \emptyset$, $Q' \cap Q^* \ne \emptyset$ and $\mathrm{diam}\, Q' \le \frac{1}{2} \mathrm{diam}\, K$. It remains to make use of the triangle inequality and the required inequality $\|x_K - x_Q\|_\infty \le (5/2)\, \mathrm{diam}\, K$ follows. $\square$

Lemma 3.8 *Let S be a regular subset of $\mathbf{R}^n$ and let $f \in L_{u,loc}(S)$, $1 \le u \le \infty$. Then for every cube K and every polynomial $P_0 \in \mathcal{P}_{k-1}$*

$$\|\tilde{f} - P_0\|^u_{L_u(K\setminus S)} \le C \sum_{Q \in \mathcal{Q}_2(K)} \|P_Q - P_0\|^u_{L_u(Q)}.$$

Proof. Clearly, $K \setminus S \subset \cup\{Q : \ Q \in \mathcal{Q}_1(K)\}$ so that

$$\|\tilde{f} - P_0\|^u_{L_u(K\setminus S)} \le \sum_{Q \in \mathcal{Q}_1(K)} \|\tilde{f} - P_0\|^u_{L_u(Q)}.$$

Let $Q \in \mathcal{Q}_1(K)$. By $V(Q)$ we denote a family of cubes

$$V(Q) := \{Q' \in W_S : (Q')^* \cap Q \ne \emptyset\}.$$

Clearly, by property (2) of Lemma 2.3 $M_Q := \operatorname{card} V(Q) \le N(n)$. Properties (a)-(c) of the partition of unity and formula (3.6) imply

$$\begin{aligned}
\|\tilde{f} - P_0\|^u_{L_u(Q)} &\le \| \sum_{Q' \in W_S} \varphi_{Q'}(P_{Q'} - P_0)\|^u_{L_u(Q)} \\
&= \| \sum_{Q' \in V(Q)} \varphi_{Q'}(P_{Q'} - P_0)\|^u_{L_u(Q)} \le M_Q^{u-1} \sum_{Q' \in V(Q)} \|P_{Q'} - P_0\|^u_{L_u(Q)}
\end{aligned}$$

so that

$$\|\tilde{f} - P_0\|^u_{L_u(Q)} \le C \sum_{Q' \in V(Q)} \|P_{Q'} - P_0\|^u_{L_u(Q)}.$$

Since $(Q')^* \cap Q \ne \emptyset$ for every $Q' \in V(Q)$, by Lemma 2.3, (1), $\operatorname{diam} Q' \approx \operatorname{diam} Q$. Hence by Corollary 3.3

$$\|P_{Q'} - P_0\|_{L_u(Q)} \approx \|P_{Q'} - P_0\|_{L_u(Q')}$$

so that

$$\|\tilde{f} - P_0\|^u_{L_u(Q)} \le C \sum_{Q' \in V(Q)} \|P_{Q'} - P_0\|^u_{L_u(Q')}.$$

This implies

$$\|\tilde{f} - P_0\|^u_{L_u(K\setminus S)} \le C \sum_{Q \in \mathcal{Q}_1(K)} \sum_{Q' \in V(Q)} \|P_{Q'} - P_0\|^u_{L_u(Q')}.$$

Clearly, every cube Q' on the right-hand side of this inequality belongs to $\mathcal{Q}_2(K)$, see definition (3.10). Moreover, by Lemma 2.3, (2), for every such a cube Q' there are at most $N(n)$ cubes $Q \in W_S$ such that $V(Q) \ni Q'$. Hence

$$\begin{aligned}
\|\tilde{f} - P_0\|^u_{L_u(K\setminus S)} &\le C \sum_{Q' \in \mathcal{Q}_2(K)} \operatorname{card}\{Q : V(Q) \ni Q'\} \|P_{Q'} - P_0\|^u_{L_u(Q')} \\
&\le CN(n) \sum_{Q \in \mathcal{Q}_2(K)} \|P_Q - P_0\|^u_{L_u(Q)}. \qquad \square
\end{aligned}$$

Given a cube $K \subset \mathbf{R}^n$, define a family of cubes

$$\mathcal{Q}_3(K) := \{Q \in \mathcal{Q}_2(K) : \ \operatorname{diam} Q \le \delta_S\}. \tag{3.11}$$

Lemma 3.9 *Let S be a regular subset of $\mathbf{R}^n$ and let $f \in L_{u,loc}(S)$, $1 \le u \le \infty$. Then for every cube K centered in S and every polynomial $P_0 \in \mathcal{P}_{k-1}$*

$$\sum_{Q \in \mathcal{Q}_3(K)} \|P_Q - P_0\|^u_{L_u(Q)} \le C\|f - P_0\|^u_{L_u((25K)\cap S)}.$$

Proof. For each $Q \in \mathcal{Q}_3(K)$ by properties (i),(ii) of Theorem 2.4 and by Corollary 3.3 we have

$$\|P_Q - P_0\|_{L_u(Q)} \le C\|P_Q - P_0\|_{L_u(H_Q)}.$$

By Corollary 3.5

$$\|P_Q - P_0\|_{L_u(H_Q)} \le \|P_Q - f\|_{L_u(H_Q)} + \|f - P_0\|_{L_u(H_Q)} \le \gamma E_k(f; H_Q)_{L_u} + \|f - P_0\|_{L_u(H_Q)}$$

where $\gamma = \gamma(n, k, \theta_S)$. Since $E_k(f; H_Q)_{L_u} \le \|f - P_0\|_{L_u(H_Q)}$, see definition (3.2), we have

$$\|P_Q - P_0\|_{L_u(Q)} \le C\|f - P_0\|_{L_u(H_Q)}.$$

Put $B := \cup\{H_Q : Q \in \mathcal{Q}_3(K)\}$ and $\eta := \sum\{\chi_{H_Q} : Q \in \mathcal{Q}_3(K)\}$. Then the last inequality imply

$$\sum_{Q \in \mathcal{Q}_3(K)} \|P_Q - P_0\|^u_{L_u(Q)} \le C \sum_{Q \in \mathcal{Q}_3(K)} \|f - P_0\|^u_{L_u(H_Q)} = C\|\eta\,(f - P_0)\|^u_{L_u(B)}.$$

But by property (iii) of Theorem 2.4 $\eta \le \sum\{\chi_{H_Q} : Q \in W_S\} \le \gamma(n, \theta_S)$ so that

$$\sum_{Q \in \mathcal{Q}_3(K)} \|P_Q - P_0\|^u_{L_u(Q)} \le C\|f - P_0\|^u_{L_u(B)}.$$

By Lemma 3.7 for each $Q \in \mathcal{Q}_3 \subset \mathcal{Q}_2$ we have $\|x_K - x_Q\|_\infty \le (5/2)\operatorname{diam} K$ and $\operatorname{diam} Q \le 2\operatorname{diam} K$. Moreover, by property (i) of Theorem 2.4, $H_Q \subset (10Q) \cap S$. Hence

$$H_Q \subset 10Q \subset (10 \cdot 2 + 5)K = 25K$$

proving that $B \subset (25K) \cap S$. $\square$

Proposition 3.10 *Let $f \in L_{u,loc}(S)$, $1 \le u \le \infty$, where S is a regular set. Then for every cube K with $\operatorname{diam} K \le \delta_S/2$ centered in S and every polynomial $P_0 \in \mathcal{P}_{k-1}$*

$$\|\tilde{f} - P_0\|_{L_u(K)} \le C\|f - P_0\|_{L_u((25K)\cap S)}.$$

Proof. By Lemma 3.8

$$\|\tilde{f} - P_0\|^u_{L_u(K\setminus S)} \le C \sum_{Q \in \mathcal{Q}_2(K)} \|P_Q - P_0\|^u_{L_u(Q)}.$$

Since $\operatorname{diam} K \le \delta_S/2$, by Lemma 3.7 for every $Q \in \mathcal{Q}_2(K)$ we have $\operatorname{diam} Q \le 2\operatorname{diam} K$ so that $\operatorname{diam} Q \le \delta_S$. Hence $\mathcal{Q}_2(K) = \mathcal{Q}_3(K)$, see definition (3.11).

Therefore by Lemma 3.9

$$\sum_{Q\in\mathcal{Q}_2(K)}\|P_Q-P_0\|^u_{L_u(Q)}=\sum_{Q\in\mathcal{Q}_3(K)}\|P_Q-P_0\|^u_{L_u(Q)}\le C\|f-P_0\|^u_{L_u((25K)\cap S)}$$

so that

$$\|\tilde f-P_0\|^u_{L_u(K\setminus S)}\le C\|f-P_0\|^u_{L_u((25K)\cap S)}.$$

Finally,

$$\|\tilde f-P_0\|^u_{L_u(K)}=\|\tilde f-P_0\|^u_{L_u(K\cap S)}+\|\tilde f-P_0\|^u_{L_u(K\setminus S)}\le (C+1)\|f-P_0\|^u_{L_u((25K)\cap S)}$$

proving the lemma. $\square$

Let us put $P_0\in\mathcal{P}_{k-1}$ to be a polynomial of the best approximation of f on $(25K)\cap S$ in L_u-norm. Then the above proposition implies the following inequality

$$E_k(\tilde f;K)_{L_u}\le CE_k(f;(25K)\cap S)_{L_u}.$$

Since $|K|\approx|25K|$, we obtain the next

Theorem 3.11 *Let S be a regular set and let $f\in L_{u,loc}(S)$, $1\le u\le\infty$. Then for every cube K with* $\operatorname{diam}K\le\delta_S/2$ *centered in S*

$$\mathcal{E}_k(\tilde f;K)_{L_u}\le C\mathcal{E}_k(f;25K)_{L_u(S)}.$$

Let us estimate the L_u-norm of the extension $\tilde f$.

Proposition 3.12 *Let $f\in L_{u,loc}(S)$, $1\le u\le\infty$, where S is regular. Then for every cube K centered in S*

$$\|\tilde f\|_{L_u(K)}\le C\|f\|_{L_u((25K)\cap S)}.$$

Proof. By Lemma 3.8 with $P_0:=0$ we have

$$\|\tilde f\|^u_{L_u(K\setminus S)}\le C\sum_{Q\in\mathcal{Q}_2(K)}\|P_Q\|^u_{L_u(Q)}.$$

Recall that $P_Q:=0$ if $\operatorname{diam}Q>\delta_S$, see (3.5), so that by definition (3.11)

$$\|\tilde f\|^u_{L_u(K\setminus S)}\le C\sum_{Q\in\mathcal{Q}_3(K)}\|P_Q\|^u_{L_u(Q)}.$$

Now by Lemma 3.9 (with $P_0=0$) we obtain

$$\sum_{Q\in\mathcal{Q}_3(K)}\|P_Q\|^u_{L_u(Q)}\le C\|f\|^u_{L_u((25K)\cap S)}$$

so that $\|\tilde f\|_{L_u(K\setminus S)}\le C\|f\|_{L_u((25K)\cap S)}$. Finally,

$$\|\tilde f\|^u_{L_u(K)}=\|\tilde f\|^u_{L_u(K\cap S)}+\|\tilde f\|^u_{L_u(K\setminus S)}\le (C+1)\|f\|^u_{L_u((25K)\cap S)}.\qquad\square$$

We turn to estimates of local approximations of $\tilde f$ on cubes which are located rather far from the set S. In the remaining part of the section we will assume that a cube $K=Q(x_K,r_K)$ satisfies the inequality

$$\operatorname{diam}K\le\operatorname{dist}(x_K,S)/40.\tag{3.12}$$

We let $Q_K\in W_K$ denote a Whitney's cube which contains center of K, the point x_K.

Lemma 3.13 *$K \subset Q_K^*$ and*

$$\frac{1}{5}\operatorname{dist}(x_K, S) \le \operatorname{diam} Q_K \le \operatorname{dist}(x_K, S).$$

Proof. Since $x_K \in Q_K$, by Theorem 2.2, (ii),

$$\operatorname{diam} Q_K \le \operatorname{dist}(Q_K, S) \le \operatorname{dist}(x_K, S).$$

Applying again property (ii) of Theorem 2.2, we obtain

$$\operatorname{dist}(x_K, S) \le \operatorname{diam} Q_K + \operatorname{dist}(Q_K, S) \le 5 \operatorname{diam} Q_K.$$

This inequality and (3.12) imply

$$\operatorname{diam} K \le \frac{1}{40}\operatorname{dist}(x_K, S) \le \frac{1}{8}\operatorname{diam} Q.$$

Since $Q_K \cap K \ne \emptyset$, we obtain the required inclusion $K \subset (1 + \frac{1}{8})Q_K =: Q_K^*$. $\square$

Lemma 3.14 *(Brudnyi [7]) Let Q be a cube in $\mathbf{R}^n$ and let $g \in C^\infty(Q)$. Then for every $1 \le u \le \infty$*

$$\mathcal{E}_k(g; Q)_{L_u} \le C(\operatorname{diam} Q)^k \max_{|\alpha|=k} \|D^\alpha g\|_{L_\infty(Q)}.$$

Lemma 3.15 *For every cube K satisfying (3.12) and every $1 \le u \le \infty$ we have*

$$\mathcal{E}_k(\tilde{f}; K)_{L_u} \le C\left(\frac{\operatorname{diam} K}{\operatorname{dist}(x_K, S)}\right)^k \max\{\|P_Q - P_{Q_K}\|_{L_\infty(Q)} : \ Q^* \cap K \ne \emptyset\}.$$

Proof. Since $K \subset \mathbf{R}^n \setminus S$, $\tilde{f}|_K \in C^\infty(K)$, so that by Lemma 3.14

$$\mathcal{E}_k(\tilde{f}; K)_{L_u} \le C(\operatorname{diam} K)^k \max_{|\alpha|=k} \|D^\alpha \tilde{f}\|_{L_\infty(K)}.$$

Since $K \subset Q_K^*$, see Lemma 3.13, by properties of partition of unity and by Leibnitz's formula for every $|\alpha| = k$ we have

$$\begin{aligned}
\|D^\alpha \tilde{f}\|_{L_\infty(K)} &= \|D^\alpha(\sum_{Q \in W_S} \varphi_Q(P_Q - P_{Q_K}))\|_{L_\infty(K)} \\
&\le C \sum_{Q^* \cap K \ne \emptyset} \ \sum_{\alpha = \alpha_1 + \alpha_2} \|D^{\alpha_1}\varphi_Q\|_{L_\infty(K)} \cdot \|D^{\alpha_2}(P_Q - P_{Q_K})\|_{L_\infty(K)} \\
&\le C \sum_{Q^* \cap K \ne \emptyset} \ \sum_{\alpha = \alpha_1 + \alpha_2} (\operatorname{diam} Q)^{-|\alpha_1|} \|D^{\alpha_2}(P_Q - P_{Q_K})\|_{L_\infty(Q_K^*)}.
\end{aligned}$$

By Markov's inequality and Proposition 3.1

$$\begin{aligned}
\|D^{\alpha_2}(P_Q - P_{Q_K})\|_{L_\infty(Q_K^*)} &\le C(\operatorname{diam} Q_K^*)^{-|\alpha_2|} \|P_Q - P_{Q_K}\|_{L_\infty(Q_K^*)} \\
&\le C(\operatorname{diam} Q_K)^{-|\alpha_2|} \|P_Q - P_{Q_K}\|_{L_\infty(Q_K)}
\end{aligned}$$

so that

$$\mathcal{E}_k(\tilde{f};K)_{L_u} \le C\left(\frac{\operatorname{diam} K}{\operatorname{diam} Q_K}\right)^k \sum\{\|P_Q - P_{Q_K}\|_{L_\infty(Q)} :\ Q^*\cap K \ne \emptyset\}.$$

But by Lemma 3.13 $\operatorname{diam} Q_K \approx \operatorname{dist}(x_K, S)$, and the result follows. $\square$

Recall that a_{x_K} stands for a point nearest to x_K on S. Denote

$$\widetilde{Q}_K := Q(a_{x_K}, 2\operatorname{dist}(x_K, S)).$$

Then inequality (3.12) immediately implies that $\widetilde{Q}_K \supset K$.

We put

$$\mathcal{A}_K := \{Q \in \mathcal{Q}_2(\widetilde{Q}_K) :\ \ \operatorname{diam} Q \le 4\operatorname{dist}(x_K, S)\}.$$

Lemma 3.16 *For every cube K satisfying (3.12) and for every polynomial $P_0 \in \mathcal{P}_{k-1}$*

$$\mathcal{E}_k(\tilde{f};K)^u_{L_u} \le C\left(\frac{\operatorname{diam} K}{\operatorname{dist}(x_K, S)}\right)^{ku} |\widetilde{Q}_K|^{-1} \sum_{Q\in\mathcal{A}_K} \|P_Q - P_0\|^u_{L_u(Q)}.$$

Proof. For each $Q \in W_S$ such that $Q^* \cap K \ne \emptyset$ we have

$$\|P_Q - P_{Q_K}\|_{L_\infty(Q)} \le \|P_Q - P_0\|_{L_\infty(Q)} + \|P_{Q_K} - P_0\|_{L_\infty(Q)}.$$

Since $Q_K^* \supset K$, see Lemma 3.13, we have $Q^* \cap Q_K^* \ne \emptyset$ so that by Lemma 2.3, (1), $\operatorname{diam} Q \approx \operatorname{diam} Q_K$. Then by Corollary 3.3

$$\|P_{Q_K} - P_0\|_{L_\infty(Q)} \le C\|P_{Q_K} - P_0\|_{L_\infty(Q_K)}.$$

In turn, by Corollary 3.2

$$\|P_Q - P_0\|^u_{L_\infty(Q)} \le C|Q|^{-1}\|P_Q - P_0\|^u_{L_u(Q)} \le C|\widetilde{Q}_K|^{-1}\|P_Q - P_0\|^u_{L_u(Q)}.$$

Hence

$$\max\{\|P_Q - P_{Q_K}\|^u_{L_\infty(Q)} : Q^*\cap K \ne \emptyset\} \le C|\widetilde{Q}_K|^{-1}\max\{\|P_Q - P_0\|^u_{L_u(Q)} : Q^*\cap K \ne \emptyset\}$$

so that by Lemma 3.15

$$\begin{aligned}\mathcal{E}_k(\tilde{f};K)^u_{L_u} &\le C\left(\frac{\operatorname{diam} K}{\operatorname{dist}(x_K, S)}\right)^{ku} \max\{\|P_Q - P_{Q_K}\|^u_{L_\infty(Q)} :\ Q^*\cap K \ne \emptyset\}\\ &\le C\left(\frac{\operatorname{diam} K}{\operatorname{dist}(x_K, S)}\right)^{ku} |\widetilde{Q}_K|^{-1}\max\{\|P_Q - P_0\|^u_{L_u(Q)} :\ Q^*\cap K \ne \emptyset\}.\end{aligned}$$

Since $K \subset \widetilde{Q}_K$, by definition of the family $\mathcal{Q}_2$, see (3.10), every $Q \in W_S$ such that $Q^* \cap K \ne \emptyset$ belongs to $\mathcal{Q}_2(\widetilde{Q}_K)$. Moreover, by Lemma 2.3, (1), $\operatorname{diam} Q \le 4\operatorname{diam} Q_K$ and by Lemma 3.13 $\operatorname{diam} Q_K \le \operatorname{dist}(x_K, S)$. Hence $\operatorname{diam} Q \le 4\operatorname{dist}(x_K, S)$ proving that $Q \in \mathcal{A}_K$. This shows that the latter maximum can be taken over family $\mathcal{A}_K$. The the lemma is proved. $\square$

We put

$$\overline{Q}_K := 25\widetilde{Q}_K = Q(a_{x_K}, 50\operatorname{dist}(x_K, S)). \tag{3.13}$$

Lemma 3.17 *Suppose that a cube K satisfies (3.12) and* $\operatorname{dist}(x_K, S) \le \delta_S/4$. *Then*

$$\mathcal{E}_k(\tilde{f}; K)_{L_u} \le C \left(\frac{\operatorname{diam} K}{\operatorname{dist}(x_K, S)}\right)^k \mathcal{E}_k(f; \overline{Q}_K)_{L_u(S)}.$$

Proof. Since $\operatorname{dist}(x_K, S) \le \delta_S/4$,

$$\begin{aligned} \mathcal{A}_K &:= \{Q \in \mathcal{Q}_2(\widetilde{Q}_K): \ \operatorname{diam} Q \le 4 \operatorname{dist}(x_K, S)\} \\ &\subset \{Q \in \mathcal{Q}_2(\widetilde{Q}_K): \ \operatorname{diam} Q \le \delta_S\} =: \mathcal{Q}_3(\widetilde{Q}_K), \end{aligned}$$

see (3.11). Hence by Lemma 3.16 for every $P_0 \in \mathcal{P}_{k-1}$ we have

$$\mathcal{E}_k(\tilde{f}; K)^u_{L_u} \le C \left(\frac{\operatorname{diam} K}{\operatorname{dist}(x_K, S)}\right)^{ku} |\widetilde{Q}|^{-1} \sum_{Q \in \mathcal{Q}_3(\widetilde{Q}_K)} \|P_Q - P_0\|^u_{L_u(Q)}.$$

By Lemma 3.9

$$\sum_{Q \in \mathcal{Q}_3(\widetilde{Q}_K)} \|P_Q - P_0\|^u_{L_u(Q)} \le C \|f - P_0\|^u_{L_u((25\widetilde{Q}_K)\cap S)} = C\|f - P_0\|^u_{L_u(\overline{Q}_K \cap S)}.$$

It remains to put $P_0 \in \mathcal{P}_{k-1}$ to be a polynomial of the best approximation of f on $\overline{Q}_K \cap S$ in L_u-norm and the lemma follows. □

The last auxiliary result of the section is the following

Lemma 3.18 *For every cube K satisfying (3.12)*

$$\mathcal{E}_k(\tilde{f}; K)_{L_u} \le C \left(\frac{\operatorname{diam} K}{\operatorname{dist}(x_K, S)}\right)^k |\overline{Q}_K|^{-\frac{1}{u}} \|f\|_{L_u(\overline{Q}_K \cap S)}.$$

Proof. Recall that $P_Q := 0$ if $\operatorname{diam} Q > \delta_S$ so that

$$\sum_{Q \in \mathcal{Q}_2(\widetilde{Q}_K)} \|P_Q\|^u_{L_u(Q)} = \sum_{Q \in \mathcal{Q}_3(\widetilde{Q}_K)} \|P_Q\|^u_{L_u(Q)}.$$

By Lemma 3.16 with $P_0 = 0$ we obtain

$$\mathcal{E}_k(\tilde{f}; K)^u_{L_u} \le C \left(\frac{\operatorname{diam} K}{\operatorname{dist}(x_K, S)}\right)^{ku} |\widetilde{Q}_K|^{-1} \sum_{Q \in \mathcal{Q}_3(\widetilde{Q}_K)} \|P_Q\|^u_{L_u(Q)}.$$

Hence by Lemma 3.9 (with $P_0 = 0$) we have

$$\mathcal{E}_k(\tilde{f}; K)^u_{L_u} \le C \left(\frac{\operatorname{diam} K}{\operatorname{dist}(x_K, S)}\right)^{ku} |\widetilde{Q}_K|^{-1} \|f\|^u_{L_u((25\widetilde{Q}_K)\cap S)}$$

which implies the lemma because $\overline{Q}_K := 25\widetilde{Q}_K$. □

We are in a position to finish the proof of Theorem 3.6. Let us fix $x \in \mathbf{R}^n$ and $t > 0$ and consider four cases.

Case 1. $80t \le \operatorname{dist}(x,S)$ and $r^{(x,t)} \le \delta_S$. Recall that $r^{(x,t)} := 50\max\{80t, \operatorname{dist}(x,S)\}$ so that in our case $r^{(x,t)} = 50\operatorname{dist}(x,S)$. In turn,

$$K^{(x,t)} := Q(a_x, r^{(x,t)}) = Q(a_x, 50\operatorname{dist}(x,S)),$$

see (3.7) and (3.8).

Put $K := Q(x,t)$. Then $\operatorname{diam} K = 2t$ (recall that we measure distances in the uniform norm) so that $\operatorname{diam} K \le \operatorname{dist}(x,S)/40$. Moreover,

$$r^{(x,t)} = 50\operatorname{dist}(x,S) \le \delta_S$$

which, in particular, implies that $\operatorname{dist}(x,S) \le \delta_S/2$. Thus K satisfies conditions of Lemma 3.17. By this lemma

$$\mathcal{E}_k(\tilde{f};K)_{L_u} \le C\left(\frac{t}{\operatorname{dist}(x,S)}\right)^k \mathcal{E}_k(f;\overline{Q}_K)_{L_u(S)}$$

where $\overline{Q}_K := Q(a_x, 50\operatorname{dist}(x,S)) = K^{(x,t)}$, see (3.13). Since $80t \le \operatorname{dist}(x,S)$, we have $\operatorname{dist}(x,S)^k \approx t^k + \operatorname{dist}(x,S)^k$ proving Theorem 3.6 in the case under consideration.

Case 2. $80t \le \operatorname{dist}(x,S)$ and $r^{(x,t)} > \delta_S$.

We treat this case in the same way as the previous one. The only difference is we apply Lemma 3.18 rather than Lemma 3.17.

Case 3. $80t > \operatorname{dist}(x,S)$ and $r^{(x,t)} \le \delta_S$. In this case $r^{(x,t)} = 50 \cdot 80t = 4000t$ so that $4000t \le \delta_S$. Recall that $\|a_x - x\|_\infty = \operatorname{dist}(x,S)$ so that

$$K = Q(x,t) \subset Q(a_x, \operatorname{dist}(x,S)+t) \subset Q(a_x, 81t).$$

We put $\overline{K} := Q(a_x, 81t)$ so that $K \subset \overline{K}$. Then by (3.3)

$$\mathcal{E}_k(\tilde{f};K)_{L_u} \le C\mathcal{E}_k(\tilde{f};\overline{K})_{L_u} \tag{3.14}$$

and by Theorem 3.11 $\mathcal{E}_k(\tilde{f};\overline{K})_{L_u} \le C\mathcal{E}_k(f;25\overline{K})_{L_u(S)}$. Observe that

$$25\overline{K} \subset K^{(x,t)} := Q(a_x, r^{(x,t)}) = Q(a_x, 4000t) \tag{3.15}$$

so that by (3.3) $\mathcal{E}_k(f;25\overline{K})_{L_u(S)} \le C\mathcal{E}_k(f;K^{(x,t)})_{L_u(S)}$.

It remains to note that $t^k + \operatorname{dist}(x,S)^k \approx t^k$ and Case 3 is proved.

Case 4. $80t > \operatorname{dist}(x,S)$ and $r^{(x,t)} > \delta_S$. We preserves notation of the previous case so that we assume that inequality (3.14) holds. Clearly, $\mathcal{E}_k(\tilde{f};\overline{K})_{L_u} \le |\overline{K}|^{-\frac{1}{u}}\|\tilde{f}\|_{L_u(\overline{K})}$ so that by Proposition 3.12

$$\mathcal{E}_k(\tilde{f};\overline{K})_{L_u} \le C|\overline{K}|^{-\frac{1}{u}}\|f\|_{L_u((25\overline{K})\cap S)}.$$

Combining this with (3.14) and (3.15) we obtain $\mathcal{E}_k(\tilde{f};K)_{L_u} \le C|\overline{K}|^{-\frac{1}{u}}\|f\|_{L_u(K^{(x,t)}\cap S)}$. Since $|\overline{K}| \approx |K^{(x,t)}|$ and $t^k + \operatorname{dist}(x,S)^k \approx t^k$ this proves Case 4 and the theorem. □

4. Estimates of sharp maximal functions: proofs of Theorem 1.2 and Theorem 1.3.

To formulate the main result of the section we fix parameters $s \geq 0$, $k \in \mathbf{N} \cup \{0\}$, $0 < q \leq \infty$, $1 \leq u \leq \infty$, and put $\vec{v} := (s, k, q, u)$. Given a function $f \in L_{u,loc}(S)$, we let $f^{\sharp}_{\vec{v},S}$ denote a *generalized sharp maximal function* of f on S:

$$f^{\sharp}_{\vec{v},S}(x) := \left\{ \int_0^\infty \left(\frac{\mathcal{E}_k(f; Q(x,t))_{L_u(S)}}{t^s} \right)^q \frac{dt}{t} \right\}^{\frac{1}{q}}, \qquad x \in S, \tag{4.1}$$

if $q < \infty$, and

$$f^{\sharp}_{\vec{v},S}(x) := \sup_{t>0} \frac{\mathcal{E}_k(f; Q(x,t))_{L_u(S)}}{t^s}, \qquad x \in S,$$

if $q = \infty$. We write $f^{\sharp}_{\vec{v}}$ for $f^{\sharp}_{\vec{v},\mathbf{R}^n}$.

As usual we put $M_u f(x) := (M(|f|^u)(x))^{\frac{1}{u}}$ where M is the Hardy-Littlewood maximal function

$$Mf(x) := \sup_{t>0} \frac{1}{|Q(x,t)|} \int_{Q(x,t)} |f(y)| dy.$$

We recall that by the Hardy-Littlewood-Wiener maximal inequality, see e.g. [38], for every $0 < u < p \leq \infty$ and $g \in L_p(\mathbf{R}^n)$

$$\|M_u g\|_{L_p(\mathbf{R}^n)} \leq C \|g\|_{L_p(\mathbf{R}^n)}. \tag{4.2}$$

Theorem 4.1 *Let S be a regular subset of $\mathbf{R}^n$ and let $f \in L_{u,loc}(S)$. Assume that $1 \leq u \leq \infty$, and $0 \leq s < k$ if $0 < q \leq \infty$ or $0 \leq s \leq k$ if $q = \infty$. Then*

$$(\tilde{f})^{\sharp}_{\vec{v}}(x) \leq C\{M((f^{\sharp}_{\vec{v},S})^{\curlywedge})(x) + M_u(f^{\curlywedge})(x)\}, \qquad x \in \mathbf{R}^n.$$

Recall that $\tilde{f}$ stands for the extension of f defined by formula (3.6). Recall also that $h^{\curlywedge}$ where h is a function on S denotes the extension by 0 of h from S on all of $\mathbf{R}^n$.
Proof. We will prove the result for $0 < q < \infty$; the reader can easy modify this proof for the case $q = \infty$. Let us consider two cases.

Case 1. We assume that

$$\operatorname{dist}(x, S) \leq \delta_S/50. \tag{4.3}$$

Put $\Delta := \delta_S/4000$. Then

$$(\tilde{f})^{\sharp}_{\vec{v}}(x) := \left\{ \int_0^\infty \left(\frac{\mathcal{E}_k(\tilde{f}; Q(x,t))_{L_u}}{t^s} \right)^q \frac{dt}{t} \right\}^{\frac{1}{q}} \leq C(I_1 + I_2)$$

where

$$I_1 := \left\{ \int_0^\Delta \left(\frac{\mathcal{E}_k(\tilde{f}; Q(x,t))_{L_u}}{t^s} \right)^q \frac{dt}{t} \right\}^{\frac{1}{q}} \text{ and } I_2 := \left\{ \int_\Delta^\infty \left(\frac{\mathcal{E}_k(\tilde{f}; Q(x,t))_{L_u}}{t^s} \right)^q \frac{dt}{t} \right\}^{\frac{1}{q}}.$$

Let us estimate I_1. We observe that for every $0 < t \le \Delta$ by inequality (4.3) the quantity $r^{(x,t)} := 50\max\{80t, \operatorname{dist}(x,S)\}$ satisfies the inequality $r^{(x,t)} \le \delta_S$. Therefore by Theorem 3.6

$$\mathcal{E}_k(\tilde{f}; Q(x,t))_{L_u} \le C \frac{t^k}{t^k + \operatorname{dist}(x,S)^k}\, \mathcal{E}_k(f; K^{(x,t)})_{L_u(S)}.$$

Recall that $K^{(x,t)} := Q(a_x, r^{(x,t)})$ where a_x is a point on S such that

$$\|x - a_x\| = \operatorname{dist}(x,S). \tag{4.4}$$

Hence

$$I_1 \le \left\{ \int_0^\Delta \left(\frac{t^{k-s}}{t^k + \operatorname{dist}(x,S)^k}\, \mathcal{E}_k(f; K^{(x,t)})_{L_u(S)} \right)^q \frac{dt}{t} \right\}^{\frac{1}{q}} \le C(J_1 + J_2)$$

where

$$J_1 := \left\{ \int\limits_0^{\operatorname{dist}(x,S)/80} \left(\frac{t^{k-s}}{\operatorname{dist}(x,S)^k}\, \mathcal{E}_k(f; K^{(x,t)})_{L_u(S)} \right)^q \frac{dt}{t} \right\}^{\frac{1}{q}}$$

and

$$J_2 := \left\{ \int\limits_{\operatorname{dist}(x,S)/80}^{\infty} \left(\frac{\mathcal{E}_k(f; K^{(x,t)})_{L_u(S)}}{t^s} \right)^q \frac{dt}{t} \right\}^{\frac{1}{q}}.$$

Prove that $J_1 \le C\, J_2$. We observe that for every $0 < t \le \operatorname{dist}(x,S)/80$ we have $r^{(x,t)} := 50\operatorname{dist}(x,S)$ so that $K^{(x,t)} := Q(a_x, 50\operatorname{dist}(x,S))$. Hence

$$J_1 \le C\, \frac{\mathcal{E}_k(f; Q(a_x, 50\operatorname{dist}(x,S)))_{L_u(S)}}{\operatorname{dist}(x,S)^k} \left\{ \int\limits_0^{\operatorname{dist}(x,S)/80} t^{(k-s)q} \frac{dt}{t} \right\}^{\frac{1}{q}}.$$

Since $k > s \ge 0$ or $k \ge s \ge 0$ if $q = \infty$, the latter integral is equivalent $\operatorname{dist}(x,S)^{(k-s)q}$. Hence

$$J_1 \le C\, \frac{\mathcal{E}_k(f; Q(a_x, 50\operatorname{dist}(x,S)))_{L_u(S)}}{\operatorname{dist}(x,S)^s}.$$

By (3.3) for every t such that $\operatorname{dist}(x,S) < t \le 2\operatorname{dist}(x,S)$ we have

$$\mathcal{E}_k(f; Q(a_x, 50\operatorname{dist}(x,S)))_{L_u(S)} \approx \mathcal{E}_k(f; Q(a_x, 50t))_{L_u(S)}$$

so that

$$\frac{\mathcal{E}_k(f; Q(a_x, 50\operatorname{dist}(x,S)))_{L_u(S)}}{\operatorname{dist}(x,S)^s} \approx \left\{ \int\limits_{\operatorname{dist}(x,S)}^{2\operatorname{dist}(x,S)} \left(\frac{\mathcal{E}_k(f; Q(a_x, 50t))_{L_u(S)}}{t^s} \right)^q \frac{dt}{t} \right\}^{\frac{1}{q}}.$$

Observe that for $t \geq \operatorname{dist}(x,S)/80$ we have $r^{(x,t)} := 50 \max\{80t, \operatorname{dist}(x,S)\} = 4000t$ and $K^{(x,t)} := Q(a_x, 4000t)$ so that

$$J_2 \approx \left\{ \int\limits_{\operatorname{dist}(x,S)}^{\infty} \left(\frac{\mathcal{E}_k(f; Q(a_x, 50t))_{L_u(S)}}{t^s} \right)^q \frac{dt}{t} \right\}^{\frac{1}{q}} \tag{4.5}$$

proving the required inequality $J_1 \leq C J_2$.

Let us estimate J_2. To this end we put $\widetilde{K} := Q(x, 2\operatorname{dist}(x,S))$. Prove that for each $y \in \widetilde{K} \cap S$ we have $J_2 \leq C f^{\sharp}_{\vec{v},S}(y)$. In fact, by (4.4)

$$\|y - a_x\| \leq \|y - x\| + \|x - a_x\| \leq 3 \operatorname{dist}(x,S).$$

Hence for every $t > 50 \operatorname{dist}(x,S)$ we have $Q(a_x, t) \subset Q(y, 2t)$. This inclusion and (3.3) imply

$$\mathcal{E}_k(f; Q(a_x, t))_{L_u(S)} \leq C \mathcal{E}_k(f; Q(y, 2t))_{L_u(S)}.$$

so that by (4.5)

$$J_2 \leq C \left\{ \int\limits_{\operatorname{dist}(x,S)}^{\infty} \left(\frac{\mathcal{E}_k(f; Q(y, 100t))_{L_u(S)}}{t^s} \right)^q \frac{dt}{t} \right\}^{\frac{1}{q}} \leq C f^{\sharp}_{\vec{v},S}(y)$$

proving the required inequality $J_2 \leq C f^{\sharp}_{\vec{v},S}(y)$. By this inequality

$$J_2 \leq C \frac{1}{|\widetilde{K} \cap S|} \int\limits_{\widetilde{K} \cap S} f^{\sharp}_{\vec{v},S}(y) dy.$$

Since $\operatorname{dist}(x,S) \geq \delta_S/50$, see (4.3), by (4.4) we have

$$Q(a_x, \operatorname{dist}(x,S)) \subset Q(x, 2\operatorname{dist}(x,S)) =: \widetilde{K}.$$

Since S is regular and $\operatorname{dist}(x,S) \leq \delta_S$,

$$|\widetilde{K} \cap S| \geq |Q(a_x, \operatorname{dist}(x,S)) \cap S| \geq \theta_S |Q(a_x, \operatorname{dist}(x,S))| \approx |\widetilde{K}|$$

so that $|\widetilde{K} \cap S| \approx |\widetilde{K}|$. Hence

$$J_2 \leq C \frac{1}{|\widetilde{K}|} \int\limits_{\widetilde{K} \cap S} f^{\sharp}_{\vec{v},S}(y) dy \leq C M(f^{\sharp}_{\vec{v},S})^{\curlywedge}(x).$$

Combining this with the estimate $J_1 \leq C J_2$ we conclude that $I_1 \leq C M(f^{\sharp}_{\vec{v},S})^{\curlywedge}(x)$.

Let us prove that $I_2 \leq C M_u(f^{\curlywedge})(x)$. We recall that $\operatorname{dist}(x,S) \leq \delta_S/50$ so that for every $t > \Delta := \delta_S/4000$

$$r^{(x,t)} := 50 \max\{80t, \operatorname{dist}(x,S)\} = 4000t > \delta_S$$

and $K^{(x,t)} := Q(a_x, r^{(x,t)}) = Q(a_x, 4000t)$. Therefore by Theorem 3.6 and (3.9)

$$\mathcal{E}_k(\tilde{f}; Q(x,t))_{L_u} \le C\left(\frac{1}{|K^{(x,t)}|}\int_{K^{(x,t)}\cap S}|f|^u dy\right)^{\frac{1}{u}}.$$

We put $\overline{K} := Q(x, 4080t)$. Since $\operatorname{dist}(x,S) \le \delta_S/50 \le 80t$, by (4.4) we have

$$K^{(x,t)} \subset Q(x, \operatorname{dist}(x,S) + 4000t) \subset \overline{K}.$$

Moreover, we also obtain an equivalence $|K^{(x,t)}| \approx |\overline{K}|$. Hence

$$\mathcal{E}_k(\tilde{f}; Q(x,t))_{L_u} \le C\left(\frac{1}{|\overline{K}|}\int_{\overline{K}\cap S}|f|^u dy\right)^{\frac{1}{u}} \le CM_u(f^{\curlywedge})(x).$$

This implies

$$I_2 := \left\{\int_{\Delta}^{\infty}\left(\frac{\mathcal{E}_k(\tilde{f}; Q(x,t))_{L_u}}{t^s}\right)^q \frac{dt}{t}\right\}^{\frac{1}{q}} \le C(M_u(f^{\curlywedge})(x))\left(\int_{\Delta}^{\infty} t^{-sq-1}dt\right)^{\frac{1}{q}}$$

proving the required $I_2 \le CM_u(f^{\curlywedge})(x)$. Finally, we obtain

$$(\tilde{f})^{\sharp}_{\vec{v}}(x) \le C\{I_1 + I_2\} \le C\{M((f^{\,\sharp}_{\vec{v},S})^{\curlywedge})(x) + M_u(f^{\curlywedge})(x)\}$$

which proves the theorem in the first case.

Case 2. $\operatorname{dist}(x,S) > \delta_S/50$. In this case $r^{(x,t)} := 50\max\{80t, \operatorname{dist}(x,S)\} > \delta_S$ so that by Theorem 3.6 and (3.9)

$$\mathcal{E}_k(\tilde{f}; Q(x,t))_{L_u} \le C\frac{t^k}{t^k + \operatorname{dist}(x,S)^k}\,|K^{(x,t)}|^{-\frac{1}{u}}\|f\|_{L_u(K^{(x,t)}\cap S)}. \tag{4.6}$$

Recall that $K^{(x,t)} := Q(a_x, r^{(x,t)})$.

Put $K' := Q(x, 2r^{(x,t)})$. Clearly, $r^{(x,t)} \ge 50\operatorname{dist}(x,S) \ge \operatorname{dist}(x,S)$ so that by (4.4)

$$K^{(x,t)} := Q(a_x, r^{(x,t)}) \subset Q(x, \operatorname{dist}(x,S) + r^{(x,t)}) \subset Q(x, 2r^{(x,t)}) =: K'.$$

Hence

$$|K^{(x,t)}|^{-\frac{1}{u}}\|f\|_{L_u(K^{(x,t)}\cap S)} \le C|K'|^{-\frac{1}{u}}\|f\|_{L_u(K'\cap S)} \le CM_u(f^{\curlywedge})(x). \tag{4.7}$$

Estimates (4.6) and (4.7) and definition (4.1) imply

$$(\tilde{f})^{\sharp}_{\vec{v}}(x) \le CM_u(f^{\curlywedge})(x)\left(\int\limits_0^{\infty}\frac{t^{(k-s)q}}{(t^k + \operatorname{dist}(x,S)^k)^q}\frac{dt}{t}\right)^{\frac{1}{q}}.$$

Since $\operatorname{dist}(x,S) \ge \delta_S/50$ and $k > s$ (or $k \ge s$ if $q = \infty$), the latter integral is bounded by a constant depending only on s, k, q and δ_S. This proves that in the case under consideration $(\tilde{f})^{\sharp}_{\vec{v}}(x) \le CM_u(f^{\curlywedge})(x)$.

Theorem 4.1 is completely proved. $\square$

Let us formulate a corollary of this result. To this end we introduce a slight generalization of the maximal function (4.1): given $\vec{v}=(s,k,q,u)$ and $0<\Delta\le\infty$, we put

$$f^{\sharp}_{\vec{v},\Delta,S}(x):=\left\{\int\limits_0^{\Delta}\left(\frac{\mathcal{E}_k(f;Q(x;t))_{L_u(S)}}{t^s}\right)^q\frac{dt}{t}\right\}^{\frac{1}{q}},\qquad x\in S,$$

(with the standard modification for $q=\infty$).

Theorem 4.2 *Suppose that $1\le u<p\le\infty$, $0<q\le\infty$ and $k>s\ge 0$ or $k\ge s\ge 0$ if $q=\infty$. If S is a regular set and $f\in L_p(S)$, then*

$$\|(\tilde{f})^{\sharp}_{\vec{v},\Delta,\mathbf{R}^n}\|_{L_p(\mathbf{R}^n)}\le C(\|f^{\sharp}_{\vec{v},\Delta,S}\|_{L_p(S)}+\|f\|_{L_p(S)}).$$

Here the constant C depends also on Δ.

Proof. Clearly, $(\tilde{f})^{\sharp}_{\vec{v},\Delta,\mathbf{R}^n}\le(\tilde{f})^{\sharp}_{\vec{v}}\,(:=(\tilde{f})^{\sharp}_{\vec{v},\infty,\mathbf{R}^n})$ so that

$$\|(\tilde{f})^{\sharp}_{\vec{v},\Delta,\mathbf{R}^n}\|_{L_p(\mathbf{R}^n)}\le\|(\tilde{f})^{\sharp}_{\vec{v}}\|_{L_p(\mathbf{R}^n)}.\tag{4.8}$$

By Theorem 4.1

$$\|(\tilde{f})^{\sharp}_{\vec{v}}\|_{L_p(\mathbf{R}^n)}\le C(\|M(f^{\sharp}_{\vec{v},S})^{\curlywedge}\|_{L_p(\mathbf{R}^n)}+\|M_u(f^{\curlywedge})\|_{L_p(\mathbf{R}^n)})$$

so that by (4.2)

$$\|(\tilde{f})^{\sharp}_{\vec{v}}\|_{L_p(\mathbf{R}^n)}\le C(\|f^{\sharp}_{\vec{v},S}\|_{L_p(S)}+\|f\|_{L_p(S)}).\tag{4.9}$$

On the other hand, for every $x\in\mathbf{R}^n$,

$$f^{\sharp}_{\vec{v},S}(x):=\left\{\int_0^{\infty}\left(\frac{\mathcal{E}_k(f;Q(x,t))_{L_u(S)}}{t^s}\right)^q\frac{dt}{t}\right\}^{\frac{1}{q}}\le C(f^{\sharp}_{\vec{v},\Delta,S}(x)+J(x))$$

where

$$J(x):=\left\{\int_{\Delta}^{\infty}\left(\frac{\mathcal{E}_k(f;Q(x,t))_{L_u(S)}}{t^s}\right)^q\frac{dt}{t}\right\}^{\frac{1}{q}}.$$

Observe that

$$\mathcal{E}_k(f;Q(x,t))_{L_u(S)}\le\left(\frac{1}{|Q(x,t)|}\int_{Q(x,t)\cap S}|f|^u dy\right)^{\frac{1}{u}}\le M_u(f^{\curlywedge})(x).$$

Hence

$$J(x)\le CM_u(f^{\curlywedge})(x)\left\{\int_{\Delta}^{\infty}t^{-sq-1}dt\right\}^{\frac{1}{q}}\le C\Delta^{-sq}M_u(f^{\curlywedge})(x).$$

Thus $f^{\sharp}_{\vec{v},S}(x)\le C(f^{\sharp}_{\vec{v},\Delta,S}(x)+M_u(f^{\curlywedge})(x))$ so that

$$\|f^{\sharp}_{\vec{v},S}\|_{L_p(S)}\le C(\|f^{\sharp}_{\vec{v},\Delta,S}\|_{L_p(S)}+\|M_u(f^{\curlywedge})\|_{L_p(S)})\le C(\|f^{\sharp}_{\vec{v},\Delta,S}\|_{L_p(S)}+\|f\|_{L_p(S)}).$$

This inequality, (4.8) and (4.9) imply the statement of the theorem. $\square$

P r o o f s of Theorems 1.2 and 1.3.

Observe that for every locally integrable extension F of f on all of $\mathbf{R}^n$ and for each cube Q centered in S we have $\mathcal{E}_k(f;Q)_{L_1(S)} \le \mathcal{E}_k(F;Q)_{L_1}$ so that $f^\sharp_{k,S} \le F^\sharp_k$ on S. Then by (1.3)

$$\|f\|_{L_p(S)} + \|f^\sharp_{k,S}\|_{L_p(S)} \le \|F\|_{L_p(\mathbf{R}^n)} + \|F^\sharp_k\|_{L_p(\mathbf{R}^n)} \le C\|F\|_{W^k_p(\mathbf{R}^n)}$$

proving that

$$\|f\|_{L_p(S)} + \|f^\sharp_{k,S}\|_{L_p(S)} \le C\|f\|_{W^k_p(\mathbf{R}^n)|_S}.$$

In a similar way using equivalence (1.6) we show that

$$\|f\|_{L^p(S)} + \left\| \left(\int_0^1 \left(\frac{\mathcal{E}_k(f;Q(\cdot,t))_{L_1(S)}}{t^s} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \right\|_{L_p(S)} \le C\|f\|_{F^s_{pq}(\mathbf{R}^n)|_S}.$$

To prove the opposite inequalities we observe that by Proposition 3.12 we have $\|\tilde{f}\|_{L_p(\mathbf{R}^n)} \le C\|f\|_{L_p(S)}$. Moreover, by Theorem 4.2 with $1 < p \le \infty, k = s, q = \infty, u = 1$ and $\Delta = \infty$

$$\|(\tilde{f})^\sharp_k\|_{L^p(\mathbf{R}^n)} \le C(\|f^\sharp_{k,S}\|_{L^p(S)} + \|f\|_{L_p(S)}).$$

Hence

$$\|f\|_{W^k_p(\mathbf{R}^n)|_S} \le \|\tilde{f}\|_{W^k_p(\mathbf{R}^n)} \le C(\|\tilde{f}\|_{L_p(\mathbf{R}^n)} + \|(\tilde{f})^\sharp_k\|_{L_p(\mathbf{R}^n)}) \le C(\|f\|_{L_p(S)} + \|f^\sharp_{k,S}\|_{L^p(S)})$$

proving (1.5). In a similar way we prove equivalence (1.7) applying Theorem 4.2 with $0 < s < k$, $1 < p \le \infty$, $1 \le q \le \infty, u = 1$ and $\Delta = 1$. $\square$

5. Besov spaces on regular subsets of $\mathbf{R}^n$.

We turn to the problem of an intrinsic characterization of traces of the Besov spaces to regular subsets of $\mathbf{R}^n$. First we recall one of the equivalent definitions of the Besov spaces: a function f defined on $\mathbf{R}^n$ belongs to the space $B^s_{pq}(\mathbf{R}^n)$, $1 \le p \le \infty, 0 < q \le \infty$, $s > 0$, if $f \in L_p(\mathbf{R}^n)$ and its modulus of continuity of order k in L_p

$$\omega_k(f;t)_{L_p} := \sup_{\|h\| \le t} \|\triangle^k_h f\|_{L_p(\mathbf{R}^n)}$$

satisfies the inequality

$$\int_0^1 \left(\frac{\omega_k(f;t)_{L_p}}{t^s} \right)^q \frac{dt}{t} < \infty$$

$(\sup\limits_{0<t\le 1} t^{-s}\omega_k(f,t)_{L_p} < \infty$ if $q = \infty)$. Here $k > s$ is an arbitrary integer and as usual given $x, h \in \mathbf{R}^n$,

$$\triangle^k_h f(x) := \sum_{j=0}^k (-1)^{k-j} \binom{n}{j} f(x + jh).$$

$B^s_{pq}(\mathbf{R}^n)$ is normed by

$$\|f\|_{B^s_{pq}(\mathbf{R}^n)} := \|f\|_{L_p(\mathbf{R}^n)} + \left(\int_0^1 \left(\frac{\omega_k(f;t)_{L_p}}{t^s} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \tag{5.1}$$

(modification if $q = \infty$).

Similar to the case of Sobolev and F-spaces the main point of our approach to intrinsic characterization of traces of Besov spaces is local approximations theory.

As we have mentioned above this theory gives a unified approach to various types of function spaces based on the concept of local best approximation by polynomials, see definitions (1.1) and (3.2). Comparing classical approximation theory and local approximation theory we observe that one basic goal of classical approximation theory is to study functions via the behavior of their best approximations as a function of the *degree* of the approximating polynomials on a *fixed* set. In local approximation theory we have a similar goal, but rather than doing all approximations on a fixed set, we do it on a *variable* cube. We can think of it as a "window" which we can slide around, enlarge and contract, "looking" through it at the function's graph. Each time we consider approximation on the cube by polynomials of a *fixed* (maybe small) degree, and we study the behavior of the best approximations as a function of the position and size of the sliding cube.

As an important example illustrating this idea we present so-called an "atomic" decomposition of the modulus of continuity due to Brudnyi [7, 10], see also [5, 6, 9]. This basic fact of local approximation theory states that for every $0 < p \le \infty$, $k \in \mathbf{N}$, and every function $f \in L_{p,\,loc}(\mathbf{R}^n)$

$$\omega_k(f;t)_{L_p} \approx \sup_\pi \left\{ \sum_{Q\in\pi} E_k(f;Q)^p_{L_p} \right\}^{\frac{1}{p}} \tag{5.2}$$

where the supremum is taken over all *packings* π of equal cubes in $\mathbf{R}^n$ with diameter t. (Hereafter "packing" means a *finite family of disjoint cubes* in $\mathbf{R}^n$.) Observe that equivalence (5.2) remains true if π runs over all packings of equal cubes with diameter *at most* t, see [7].

This result motivates the following definition, see [9]: given $k \in \mathbf{N}$, $0 < u, p \le \infty$, and a function $f \in L_{u,\,loc}$, by $\Omega_{k,p}(f;\cdot)_{L_u}$ we denote the *(k,p)-modulus of continuity of f in L_u*, i.e., a function of $t > 0$ defined by the following formula

$$\Omega_{k,p}(f;t)_{L_u} := \sup_\pi \left\{ \sum_{Q\in\pi} |Q|\, \mathcal{E}_k(f;Q)^p_{L_u} \right\}^{\frac{1}{p}}. \tag{5.3}$$

Here π runs over all packings of equal cubes in $\mathbf{R}^n$ with diameter t. (This definition is a slight modification of that given in [7] where the supremum is taken over all packings π of equal cubes with diameter *at most* t.)

We note two important properties of the (k,p)-modulus of continuity. First of them is the following equivalence, see [9], Chapter 3, and Lemma 5.2:

$$\Omega_{k,p}(f;t)_{L_u} \approx \|\mathcal{E}_k(f;Q(\cdot;t))_{L_u}\|_{L_p(\mathbf{R}^n)}. \tag{5.4}$$

In particular, from (5.4) and (5.2) it follows that

$$\omega_k(f;t)_{L_p} \approx \Omega_{k,p}(f;t)_{L_p} \approx \|\mathcal{E}_k(f;Q(\cdot;t))_{L_p}\|_{L_p(\mathbf{R}^n)}, \qquad t>0. \tag{5.5}$$

The second property clarifies connections between the (k,p)-moduli of continuity in different metrics. Clearly, $\Omega_{k,p}(f;\cdot)_{L_u} \le \Omega_{k,p}(f;t)_{L_p}$ whenever $0<u\le p$. On the other hand, Brudnyi [9, 7] has proved that for every $1 \le u \le p$

$$\Omega_{k,p}(f;t)_{L_p} \le C\int_0^t \frac{\Omega_{k,p}(f;\tau)_{L_u}}{\tau}\,d\tau, \qquad t>0. \tag{5.6}$$

Now combining definition (5.1), equivalence (5.4) and inequality (5.6) and applying the Hardy inequality we obtain characterization (1.8) of Besov functions on $\mathbf{R}^n$ via local approximations.

Let us generalize definition (5.3) for the case of a measurable subset $S \subset \mathbf{R}^n$ and a function $f \in L_{u,\,loc}(S)$. We define the (k,p)-modulus of continuity of f in $L_u(S)$ ([9]) by letting

$$\Omega_{k,p}(f;t)_{L_u(S)} := \sup_{\pi}\left\{\sum_{Q\in\pi} |Q\cap S|\,\mathcal{E}_k(f;Q)^p_{L_u(S)}\right\}^{\frac{1}{p}}. \tag{5.7}$$

Here π runs over all packings of equal cubes centered in S with diameter t.

Let us show that an analog of equivalence (5.4) is true for $\Omega_{k,p}(f;\cdot)_{L_u(S)}$ as well. To prove this we need the following simple combinatorial lemma.

Lemma 5.1 *Let π be a family of equal cubes such that $\sum\{\chi_Q:\ Q\in\pi\} \le l$ where l is a positive constant. Then a family of cubes $\{2Q:\ Q\in\pi\}$ can be represented as union of at most $m=m(l)$ packings.*

In particular, from the lemma and definition (5.7) it easily follows that $\Omega_{k,p}(f;\cdot)_{L_u(S)}$ is a quasi-monotone function, i.e.,

$$\Omega_{k,p}(f;t)_{L_u(S)} \le C\,\Omega_{k,p}(f;2t)_{L_u(S)}, \qquad t>0. \tag{5.8}$$

Lemma 5.2 *Let $0<u,p\le\infty$ and $k\in\mathbf{N}$. Then for every function $f \in L_{u,\,loc}(S)$*

$$\frac{1}{C}\,\Omega_{k,p}(f;t/4)_{L_u(S)} \le \|\mathcal{E}_k(f;Q(\cdot,t))_{L_u(S)}\|_{L_p(S)} \le C\,\Omega_{k,p}(f;t)_{L_u(S)}, \qquad t>0.$$

Proof. We will mainly follow a scheme of the proof given in [9] for the case $S=\mathbf{R}^n$. Fix $t>0$ and consider a packing π of equal cubes with diameter t centered in S. Then for each $Q\in\pi$ and every $x\in Q\cap S$ we have $Q\subset Q(x,4t)$ so that by (3.3)

$$\mathcal{E}_k(f;Q)_{L_u(S)} \le C\mathcal{E}_k(f;Q(x,4t))_{L_u(S)}.$$

Hence

$$|Q\cap S|\,\mathcal{E}_k(f;Q)^p_{L_u(S)} \le C\int_{Q\cap S} \mathcal{E}_k(f;Q(x,4t))^p_{L_u(S)}dx, \qquad x\in Q\cap S.$$

Thus

$$\sum_{Q\in\pi}|Q\cap S|\,\mathcal{E}_k(f;Q)^p_{L_u(S)} \le C\sum_{Q\in\pi}\int_{Q\cap S}\mathcal{E}_k(f;Q(x,4t))^p_{L_u(S)}dx \le C\int_S \mathcal{E}_k(f;Q(x,4t))^p_{L_u(S)}dx$$

proving the first inequality of the lemma.

To prove the second inequality given $t>0$, we let $\tilde\pi$ denote a covering of S by equal cubes centered in S with diameter $t/2$ such that $\sum\{\chi_Q:\ Q\in\tilde\pi\}\le C(n)$. (The existence of $\tilde\pi$ immediately follows, for instance, from the Besicovitch theorem, see e.g. Gusman [23].) Then

$$\int_S \mathcal{E}_k(f;Q(x,t))^p_{L_u(S)}dx \le \sum_{Q\in\tilde\pi}\int_{Q\cap S}\mathcal{E}_k(f;Q(x,t))^p_{L_u(S)}dx.$$

Clearly, for every $Q\in\tilde\pi$ and every $x\in Q\cap S$ we have $Q(x,t)\subset 2Q=Q(x_Q,2t)$ so that by (3.3) $\mathcal{E}_k(f;Q(x,t))_{L_u(S)}\le C\mathcal{E}_k(f;2Q)_{L_u(S)}$. Hence

$$\int_{Q\cap S}\mathcal{E}_k(f;Q(x,t))^p_{L_u(S)}dx \le C|2Q\cap S|\,\mathcal{E}_k(f;2Q)^p_{L_u(S)}$$

so that

$$\int_S \mathcal{E}_k(f;Q(x,t))^p_{L_u(S)}dx \le C\sum_{Q\in\tilde\pi}|2Q\cap S|\,\mathcal{E}_k(f;2Q)^p_{L_u(S)}.$$

By Lemma 5.1 a family of cubes $\pi:=\{2Q:\ Q\in\tilde\pi\}$ can be represented in the form $\pi=\cup\{\pi_i:\ i=1,...,m\}$ where $m=m(n)$ and every family π_i is a packing. Hence

$$\int_S \mathcal{E}_k(f;Q(x,t))^p_{L_u(S)}dx \le C\sum_{i=1}^m\sum_{Q\in\pi_i}|Q\cap S|\,\mathcal{E}_k(f;Q)^p_{L_u(S)} \le Cm\,\Omega_{k,p}(f;t)_{L_u(S)}$$

proving the lemma. □

The main result of the section is the following

Theorem 5.3 *Let S be a regular set and let $1\le u\le p\le\infty$. Then for every function $f\in L_{u,loc}(S)$ and every $0<t\le 1$*

$$\|\mathcal{E}_k(\tilde f;Q(\cdot,t))_{L_u}\|_{L_p(\mathbf{R}^n)} \le C\,t^k\left\{\left(\int_t^1\left(\frac{\|\mathcal{E}_k(f;Q(\cdot,\tau))_{L_u(S)}\|_{L_p(S)}}{\tau^k}\right)^p\frac{d\tau}{\tau}\right)^{\frac{1}{p}}+\|f\|_{L_p(S)}\right\}$$

Here $\tilde f=\mathrm{Ext}_{k,u,S}$ is the extension operator defined by formula (3.6).

Proof. By Lemma 5.2 it is sufficient to show that for every $0<t\le 1$

$$\Omega_{k,p}(\tilde f;t)_{L_u} \le C\,t^k\left\{\left(\int_{t/4}^{1/4}\left(\frac{\Omega_{k,p}(f;\tau)_{L_u(S)}}{\tau^k}\right)^p\frac{d\tau}{\tau}\right)^{\frac{1}{p}}+\|f\|_{L_p(S)}\right\}. \tag{5.9}$$

First let us estimate $\Omega_{k,p}(f;t/2)_{L_u}$ for $0 < t \le \delta_S/4000$. Fix a family π of equal cubes in $\mathbf{R}^n$ of diameter $t/2$. (Thus $Q = Q(x_Q, t)$ for every $Q \in \pi$.) We let $m = m(t)$ denote a positive integer such that

$$\frac{1}{2^{m+1}}\,\frac{\delta_S}{50} < 80t \le \frac{1}{2^m}\,\frac{\delta_S}{50}. \tag{5.10}$$

Then for each integer $i,\ i < m$ we put

$$\pi_i := \left\{Q \in \pi : \ \frac{1}{2^{i+1}}\,\frac{\delta_S}{50} < \operatorname{dist}(x_Q, S) \le \frac{1}{2^i}\,\frac{\delta_S}{50}\right\}. \tag{5.11}$$

We also set

$$\pi_m := \left\{Q \in \pi : \ \operatorname{dist}(x_Q, S) \le \frac{1}{2^m}\,\frac{\delta_S}{50}\right\}. \tag{5.12}$$

Now following formulas (3.7) and (3.8) we assign every $Q = Q(x_Q, t) \in \pi$ a number

$$r^{(x_Q,t)} := 50 \max(80t, \operatorname{dist}(x_Q, S))$$

and a cube $K^{(x_Q,t)} := Q(a_{x_Q}, r^{(x_Q,t)})$. (Recall that $a_{x_Q} \in S$ and satisfies the equality $\|x_Q - a_{x_Q}\|_\infty = \operatorname{dist}(x_Q, S)$.)

In particular, for every $Q \in \pi_i$, $0 \le i < m$, we have $\operatorname{dist}(x_Q, S) > 80t$, so that in this case

$$r^{(x_Q,t)} = 50 \operatorname{dist}(x_Q, S) \tag{5.13}$$

and $K^{(x_Q,t)} = Q(a_{x_Q}, 50 \operatorname{dist}(x_Q, S))$. In turn, for $Q \in \pi_m$ we have

$$\operatorname{dist}(x_Q, S) \le 160t \quad \text{and} \quad r^{(x_Q,t)} \approx t.$$

Moreover, for every $Q = Q(x_Q, t) \in \pi_i$, $0 \le i \le m$,

$$r^{(x_Q,t)} \le \delta_S. \tag{5.14}$$

Observe also that for each $Q = Q(x_Q, t) \in \pi_i$ with $i < 0$

$$r^{(x_Q,t)} = 50 \operatorname{dist}(x_Q, S) > \delta_S. \tag{5.15}$$

We put

$$\Omega_i := \sum_{Q \in \pi_i} |Q|\, \mathcal{E}_k(\tilde{f}; Q)^p_{L_u}. \tag{5.16}$$

Let us estimate Ω_i for $0 \le i \le m$. By Theorem 3.6, (5.13) and (5.14), for every $Q \in \pi_i$, $0 \le i < m$, we have

$$\mathcal{E}_k(\tilde{f}; Q)_{L_u} \le C \frac{t^k}{\operatorname{dist}(x, S)^k}\, \mathcal{E}_k(f; K^{(x_Q,t)})_{L_u(S)} \tag{5.17}$$

where by $K^{(x_Q,t)} := Q(a_{x_Q}, 50\,\mathrm{dist}(x_Q, S))$. We put

$$r_i := 2^{-i}\delta_S \qquad \text{and} \qquad K^{\{Q\}} := Q(a_{x_Q}, r_i).$$

Since $Q \in \pi_i$, by (5.11) $K^{(x_Q,t)} \subset K^{\{Q\}}$ and $r_i \approx \mathrm{dist}(x_Q, S)$ so that by (3.3) and by (5.17)

$$\mathcal{E}_k(\tilde{f}; Q)_{L_u} \le C \frac{t^k}{r_i^k}\, \mathcal{E}_k(f; K^{\{Q\}})_{L_u(S)}. \tag{5.18}$$

It can be also readily seen that Theorem 3.6 and (5.12) imply the same estimate for $i = m$ as well. Thus in what follows we will assume that inequality (5.18) is true for all $i = 0, ..., m$. Observe also that for each $Q \in \pi_i$, $i = 0, ..., m$, we have $Q \subset K^{\{Q\}}$.

Now fix i, $0 \le i \le m$, and put $\tilde{\pi}_i := \{K^{\{Q\}} : \ Q \in \pi_i\}$.

Then by Besicovitch's theorem there is a subfamily $\pi'_i \subset \tilde{\pi}_i$ such that:

(a) for every $K \in \tilde{\pi}_i$ there is a cube $K' \in \pi'_i$ such that $x_K \in K'$;

(b) $\sum\{\chi_{K'} : \ K' \in \pi'_i\} \le l(n)$.

Now for every cube $K' \in \pi'_i$ we put

$$\mathcal{A}_{K'} := \{K \in \tilde{\pi}_i : \ x_K \in K'\}.$$

Since $\mathrm{diam}\, K = \mathrm{diam}\, K'$ and $x_K \in K'$ for every $K \in \mathcal{A}_{K'}$, we have $K \subset 2K'$. Recall also that $Q \subset K^{\{Q\}}$ so that

$$\cup\{Q : \ K^{\{Q\}} \in \mathcal{A}_{K'}\} \subset 2K'. \tag{5.19}$$

By property (b) of π'_i and by Lemma 5.1 later on we may assume that the family of cubes $\{2K' : \ K' \in \pi'_i\}$ is a *packing*. By (3.3) for every $K \in \mathcal{A}_{K'}$ we have

$$\mathcal{E}_k(f; K)_{L_u(S)} \le C\mathcal{E}_k(f; 2K')_{L_u(S)}.$$

Combining this with (5.18) we obtain the following estimate of Ω_i, see (5.16):

$$\Omega_i \le C\frac{t^{kp}}{r_i^{kp}} \sum_{Q\in\pi_i} |Q|\mathcal{E}_k(f; K^{\{Q\}})^p_{L_u(S)} \le C\frac{t^{kp}}{r_i^{kp}} \sum_{K'\in\pi'_i} \left(\sum_{K^{\{Q\}}\in\mathcal{A}_{K'}} |Q| \right) \mathcal{E}_k(f; 2K')^p_{L_u(S)}.$$

Hence by (5.19)

$$\Omega_i \le C\frac{t^{kp}}{r_i^{kp}} \sum_{K'\in\pi'_i} |2K'|\, \mathcal{E}_k(f; 2K')^p_{L_u(S)}. \tag{5.20}$$

Recall that $\mathrm{diam}\, K' = 2r_i := 2^{-i+1}\delta_S \le 2\delta_S$ for every cube $K' \in \pi'_i$ which implies $\mathrm{diam}(\frac{1}{2}K') = \frac{1}{2}\,\mathrm{diam}\, K' \le \delta_S$. Since S is regular, we obtain

$$|2K'| = 4^n|(1/2)K'| \le 4^n\delta_S|(1/2)K') \cap S| \le 4^n\delta_S|(2K') \cap S|.$$

Hence

$$\Omega_i \le C\frac{t^{kp}}{r_i^{kp}}\sum_{K'\in\pi_i'}|(2K')\cap S|\,\mathcal{E}_k(f;2K')^p_{L_u(S)}.$$

We have assumed that the family of cubes $\{2K':\ K'\in\pi_i'\}$ is a packing (consisting of equal cubes of diameter $4r_i$). Therefore by definition (5.7) and by property (5.8) of $\Omega_{k,p}$ we have

$$\Omega_i \le C\,t^{kp}\,\frac{\Omega_{k,p}(f;4r_i)^p_{L_u(S)}}{(4r_i)^{kp}} \le C\,t^{kp}\int_{4r_i}^{8r_i}\left(\frac{\Omega_{k,p}(f;\tau)^p_{L_u(S)}}{\tau^k}\right)^p\frac{d\tau}{\tau}.$$

Summarizing these estimates for all $i=0,...,m$ we obtain

$$I_1 := \sum_{i=0}^{m}\Omega_i \le C\,t^{kp}\int_{2^{-m+2}\delta_S}^{8\delta_S}\left(\frac{\Omega_{k,p}(f;\tau)^p_{L_u(S)}}{\tau^k}\right)^p\frac{d\tau}{\tau}.$$

But by (5.10) $2^{-m}\delta_S\ge 4000t$ so that

$$I_1 \le C\,t^{kp}\int_{t/8}^{8\delta_S}\left(\frac{\Omega_{k,p}(f;\tau)^p_{L_u(S)}}{\tau^k}\right)^p\frac{d\tau}{\tau}\,,\qquad 0<t\le\delta_S/4000.$$

Let us estimate Ω_i for $i<0$. In this case by (5.15) and Theorem 3.6 for every $Q\in\pi_i$, $i<0$, we have

$$\mathcal{E}_k(\tilde f;Q)_{L_u} \le C\frac{t^k}{\operatorname{dist}(x,S)^k}\,\mathcal{E}_0(f;K^{(x_Q,t)})_{L_u(S)}\,. \tag{5.21}$$

We continue the proof following the same scheme as for the case $0\le i\le m$ but using estimate (5.21) rather than (5.17). Then we obtain an analog of estimate (5.20) in the form

$$\Omega_i := \sum_{Q\in\pi_i}|Q|\,\mathcal{E}_k(\tilde f;Q)^p_{L_u} \le C\frac{t^{kp}}{r_i^{kp}}\sum_{K'\in\pi_i'}|2K'|\,\mathcal{E}_0(f;2K')^p_{L_u(S)}.$$

Since $u\le p$, by the Hölder inequality

$$|2K'|\,\mathcal{E}_0(f;2K')^p_{L_u(S)} = |2K'|\left(\frac{1}{|2K'|}\int_{2K'\cap S}|f|^u dy\right)^{\frac{p}{u}} \le \int_{2K'\cap S}|f|^p dy.$$

Recall that the family $\{2K':K'\in\pi_i'\}$ is a packing so that

$$\Omega_i \le C\frac{t^{kp}}{r_i^{kp}}\sum_{K'\in\pi_i'}\int_{2K'\cap S}|f|^p dy \le C\frac{t^{kp}}{r_i^{kp}}\int_S|f|^p dy = C\frac{t^{kp}}{r_i^{kp}}\|f\|^p_{L_p(S)}.$$

Hence

$$I_2 := \sum_{i<0}\Omega_i \le Ct^{kp}\|f\|^p_{L_p(S)}\sum_{i<0}r_i^{-kp} = Ct^{kp}\delta_S^{-kp}\|f\|^p_{L_p(S)}\sum_{i<0}2^{ikp} \le Ct^{kp}\|f\|^p_{L_p(S)}.$$

Finally, we obtain

$$\sum_{Q\in\pi}|Q|\,\mathcal{E}_k(\tilde{f};Q)^p_{L_u}\le I_1+I_2\le C\,t^{kp}\left\{\int_{t/8}^{8\delta_S}\left(\frac{\Omega_{k,p}(f;\tau)^p_{L_u(S)}}{\tau^k}\right)^p\frac{d\tau}{\tau}+\|f\|^p_{L_p(S)}\right\}.$$

We recall that π is an arbitrary packing of equal cubes with diameter $t/2$ so that by definition (5.3) we obtain

$$\Omega_{k,p}(\tilde{f};t/2)_{L_u}\le C\,t^k\left\{\left(\int_{t/8}^{8\delta_S}\left(\frac{\Omega_{k,p}(f;\tau)^p_{L_u(S)}}{\tau^k}\right)^p\frac{d\tau}{\tau}\right)^{\frac{1}{p}}+\|f\|_{L_p(S)}\right\},\ \ 0<t\le\frac{\delta_S}{4000}.$$

To finish the proof of (5.9) we observe that for $t/2\in[\min\{8\delta_S,1/2\},8\delta_S]$ we have

$$\Omega_{k,p}(f;t)_{L_u(S)}\le C\|f\|_{L_p(S)}. \tag{5.22}$$

(This immediately follows from definition of $\Omega_{k,p}$, see (5.7), and the Hölder inequality.) This allows us to replace the upper limit in the latter integral by $1/2$ proving that inequality (5.9) is true for $0<t\le\delta_S/8000$.

It remains to note that inequality (5.22) is true for $t/2\in[\min\{\delta_S/4000,1/2\},1/2]$ as well which immediately implies that (5.9) is true on all of the segment $[0,1]$. The proof of inequality (5.9) is finished and we are done. □

Remark 5.4 *Inequality (1.11) follows from equivalence (5.5) and Theorem 5.3 with* $u=p$*.*

Proof of Theorem 1.6. Clearly, $\mathcal{E}_k(f;Q)_{L_u(S)}\le\mathcal{E}_k(F;Q)_{L_u}$ where $F\in L_{u,loc}(\mathbf{R}^n)$ is an arbitrary extension of f on all of $\mathbf{R}^n$ and Q is an arbitrary cube centered in S. Hence

$$\|\mathcal{E}_k(f;Q(\cdot,t))_{L_u(S)}\|_{L_p(S)}\le\|\mathcal{E}_k(F;Q(\cdot,t))_{L_u}\|_{L_p(\mathbf{R}^n)}$$

so that by (1.8)

$$I:=\|f\|_{L^p(S)}+\left(\int_0^1\left(\frac{\|\mathcal{E}_k(f;Q(\cdot,t))_{L_u(S)}\|_{L_p(S)}}{t^s}\right)^q\frac{dt}{t}\right)^{\frac{1}{q}}\le C\|f\|_{B^s_{pq}(\mathbf{R}^n)|_S}.$$

Let us prove the opposite inequality. Using Theorem 5.3 and the Hardy inequality we obtain

$$J:=\left(\int_0^1\left(\frac{\|\mathcal{E}_k(\tilde{f};Q(\cdot,t))_{L_u}\|_{L_p(S)}}{t^s}\right)^q\frac{dt}{t}\right)^{\frac{1}{q}}\le C\,I.$$

We also recall that by Proposition 3.12 $\|\tilde{f}\|_{L_p(\mathbf{R}^n)}\le C\|f\|_{L_p(S)}$, so that

$$\|f\|_{B^s_{pq}(\mathbf{R}^n)|_S}\le\|\tilde{f}\|_{B^s_{pq}(\mathbf{R}^n)}\approx\|\tilde{f}\|_{L_p(\mathbf{R}^n)}+J\le C(\|f\|_{L_p(S)}+I).$$

Theorem 1.6 is proved. □

Remark 5.5 *The proof of Theorem 5.3 actually contains the following inequality: for every* $1 \le u \le p \le \infty$ *and* $f \in L_{u,loc}(S)$

$$\Omega_{k,p}(\tilde{f};t)_{L_u} \le C\,t^k \left\{ \left(\int_t^1 \left(\frac{\Omega_{k,p}(f;\tau)^p_{L_u(S)}}{\tau^k} \right)^p \frac{d\tau}{\tau} \right)^{\frac{1}{p}} + \|f\|_{L_p(S)} \right\},\ 0 < t \le 1,$$

cf. (5.9). This estimate was proved in [35], see also [36]. Using this inequality rather than the inequality of Theorem 5.3 one can prove that for $0 < s < k$, $1 \le u \le p \le \infty$ *and* $0 < q \le \infty$,

$$\|f\|_{B^s_{pq}(\mathbf{R}^n)|_S} \approx \|f\|_{L^p(S)} + \left(\int_0^1 \left(\frac{\Omega_{k,p}(f;\tau)_{L_u(S)}}{t^s} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}. \tag{5.23}$$

This version of Theorem 1.6 has been proved in [35]. For the case $1 \le p = q \le \infty$ *and* $s > 0$ *is non-integer, description (5.23) was announced in [8]; see also [25], p. 211, for another proof of this result.*

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